# ON TAKING TWISTS OF FOURIER-MUKAI TRANSFORMS

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Abstract. We show that the adjunction counit of a Fourier-Mukai transform $\Phi\colon D(X_1) \to D(X_2)$ arises from a map of the kernels of the corresponding Fourier-Mukai transforms in a very general setting of $X_{1,2}$ being proper separable schemes of finite type over a field. We write down this map of kernels explicitly – facilitating the computation of the twist (the cone of the adjunction counit) of $\Phi$. We also give another description of this map, better suited to computing cones in the case when the kernel of $\Phi$ is a pushforward from a subscheme $D \subset X_1 \times X_2$. Moreoever, we show that we can replace the condition of properness of the spaces $X_{1,2}$ by that of $D$ being proper over $X_{1,2}$ and still have this description apply as-is. This can be used, for instance, to compute spherical twists on non-proper varieties directly and in full generality.

## 1. Introduction

Let $X$ be a scheme over a field $k$ of characteristic 0. The bounded derived category $D(X)$ of coherent sheaves on $X$ has long been recognized as a crucial invariant of $X$ which holds a wealth of information about its geometry. An important class of auto-equivalences of $D(X)$, and one which comes from neither the automorphisms of $X$ nor the auto-equivalences of $\mathrm{Coh}(X)$, is formed by *spherical twists,* that is - twists by *spherical objects* or, more generally, *spherical functors.* Spherical objects have been introduced by Seidel and Thomas in [ST01] as the objects in $D(X)$ which possess the properties which make them mirror symmetric counterparts of Lagrangian subspheres on the mirror. It is then proved that these defining properties ensure that a twist by spherical object is an auto-equivalence of $D(X)$. This notion was generalised ([Ann07]) to exact functors between triangulated categories in a way that ensures that a twist by a spherical functor again produces an auto-equivalence. Spherical objects can be thought of as spherical functors $D(\mathrm{Spec}\ k) \to D(X)$: such a functor is spherical if and only if the image in $D(X)$ of the structure sheaf of Spec $k$ (which competely determines the functor) is a spherical object.

The process of taking a twist, which produces an auto-equivalence when applied to a spherical functor, is completely general and doesn't in itself rely on the fact that the functor is spherical. The ideal definition would be the following:

**Definition 1.1.** Let $C_1$ and $C_2$ be triangulated categories and let $F$ be an exact functor from $C_1$ to $C_2$ which has a right (resp. left) adjoint $R$ (resp. $L$). Then the *right twist* (resp. the *left twist*) of $F$ is defined to be the functor $T_F\colon C_2 \to C_2$ (resp. $T^l_F\colon C_1 \to C_1$) which is the functorial cone of the adjunction counit $FR \to \mathrm{Id}_{C_2}$ (resp. $LF \to \mathrm{Id}_{C_1}$).

The problem with this definition is the well-known fact that taking cones in triangulated categories is not a functorial process. A cone of a morphism between two objects is uniquely defined (up to an isomorphism), but a cone of a morphism between two functors might not exist or might not be unique. This is usually fixed by restricting ourselves to a setting where taking cone of a morphism of functors is a well-defined process (see [Ann07], §1 for details). One standard such setting is obtained by only considering the functors which are Fourier-Mukai transforms (see below) and the morphisms of functors which come from a morphism of their Fourier-Mukai kernels. In a nutshell, this paper addresses in very high generality the issues involved in taking a twist of a Fourier-Mukai transform and provides anyone who would want to actually compute one with the tools necessary for the job.

Let $X_1$ and $X_2$ be a pair of smooth projective varieties. We have the following commutative diagram of product varieties and projection maps:

$$
\begin{tikzcd}
& & X_1 \times X_2 \times X_1 \arrow[dl, "\pi_{12}"'] \arrow[d, "\pi_{13}"] \arrow[dr, "\pi_{23}"] & & \\
& X_1 \times X_2 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & X_1 \times X_1 \arrow[dll, "\tilde{\pi}_1"] \arrow[drr, "\tilde{\pi}_2"'] & X_2 \times X_1 \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_1"] & \\
X_1 & & X_2 & & X_1
\end{tikzcd}
\tag{1.1}
$$

Let $E$ be an object of $D(X_1 \times X_2)$. The *Fourier-Mukai transform from $D(X_1)$ to $D(X_2)$ with kernel $E$* is the functor

$$\Phi_E(-) = \pi_{2*}\left(E \otimes \pi_1^*(-)\right). \tag{1.2}$$

Here and throughout the paper all the functors are derived unless mentioned otherwise. It is well-known (e.g. [BO95], Lemma 1.2) that the left adjoint of $\Phi_E$ is the Fourier-Mukai transform from $D(X_2)$ to $D(X_1)$ with kernel $E^\vee \otimes \pi_1^!(\mathcal{O}_{X_1})$ where $\pi_1^!(\mathcal{O}_{X_1}) = \pi_2^*(\omega_{X_2})[\dim X_2]$. Denote this adjoint by $\Phi_E^{\mathrm{ladj}}$. As noted originally by Mukai a composition of Fourier-Mukai transforms is again a Fourier-Mukai transform ([Muk81], Prop. 1.3). In particular, $\Phi_E^{\mathrm{ladj}}\Phi_E$ is the Fourier-Mukai transform $D(X_1) \to D(X_1)$ with the kernel

$$Q = \pi_{13*}\left(\pi_{12}^* E \otimes \pi_{23}^* E^\vee \otimes \pi_{23}^* \pi_1^!(\mathcal{O}_{X_1})\right). \tag{1.3}$$

Consider now the adjunction counit

$$\Phi_E^{\mathrm{ladj}}\Phi_E \to \mathrm{Id}_{D(X_1)} \tag{1.4}$$

The identity functor $\mathrm{Id}_{D(X_1)}$ is isomorphic to a Fourier-Mukai transform $D(X_1) \to D(X_1)$ with the kernel $\mathcal{O}_\Delta = \Delta_*(\mathcal{O}_{X_1})$ where $\Delta$ is the diagonal inclusion $X_1 \hookrightarrow X_1 \times X_1$. Generally not every morphism of Fourier-Mukai functors is induced by a morphism of their kernels, but something like the adjunction counit (1.4) one does expect to come from some morphism $Q \to \mathcal{O}_\Delta$. However, despite this fact being a common knowledge amongst the specialists and the authors were unable to locate in the literature any sufficiently general writeup of it with an explicit description of the morphism $Q \to \mathcal{O}_\Delta$ involved.

We address this issue in Section 2 of this paper where we construct explicitly a morphism $Q \to \mathcal{O}_\Delta$ such that the induced morphism of the Fourier-Mukai transforms is isomorphic to the adjunction counit (1.4). However we do this in a much more general setting than that of smooth projective varieties. Thanks to the recent advances in Grothendieck duality machinery (a good exposition is given in [Lip09]) we can work with separated schemes of finite type over a field and with derived categories $D_{qc}(-)$ of unbounded complexes with quasi-coherent cohomology. This is probably as general as it gets since we need the projection maps $\pi_i \colon X_1 \times X_2 \to X_i$ to be separated and of finite type for the Grothendieck duality to work. The main result of the section is Theorem 2.1 which states that for separated schemes $X_1$ and $X_2$ of finite type over a field with $X_2$ proper (this is still needed for the left adjoint of $\Phi_E$ to be again a Fourier-Mukai transform) and for a perfect object $E$ in $D(X_1 \times X_2)$ the adjunction counit $\Phi_E^{\mathrm{ladj}}\Phi_E \to \mathrm{Id}_{D_{qc}(X_1)}$ is induced by the morphism $Q \to \mathcal{O}_\Delta$ which is roughly (for precise statement see Theorem 2.1 itself) a composition of the following:

$$\pi_{13*}\left(\text{Derived restriction to the diagonal } X_1 \times X_2 \text{ inside } X_1 \times X_2 \times X_1\right) \tag{1.5}$$

$$\Delta_* \pi_{1*}\left(\text{The evaluation map } E \otimes E^\vee \to \mathcal{O}_{X_1 \times X_2} \text{ on } X_1 \times X_2\right) \tag{1.6}$$

$$\Delta_*\left(\text{The adjunction counit } \pi_{1*}\pi_1^!(\mathcal{O}_{X_1}) \to \mathcal{O}_{X_1}\right) \tag{1.7}$$

When $X_1$ is proper, the transforms $\Phi_E$, $\Phi_E^{ladj}$ and the morphisms (1.5)-(1.7) can be restricted to the respective subcategories $D(-)$ of the categories $D_{qc}(-)$. If moreover $X_2$ is smooth we have $\pi_1^!(\mathcal{O}_{X_1}) = \pi_2^*(\omega_{X_2})[\dim X_2]$ as before.

This addresses the issue of taking the left twist of a Fourier-Mukai transform - we can now set it to be the transform defined by the cone of the morphism $Q \to \mathcal{O}_\Delta$ above. The right twists are treated similarly (see Cor. 2.5). However anyone actually trying to use the decomposition (1.5)-(1.7) to compute the cone of $Q \to \mathcal{O}_\Delta$ may find it quite ill-suited to the task. We give an example of this in Section 3.1 where we set

$E$ to be the structure sheaf $\mathcal{O}_D$ of a complete intersection subscheme $D$ in $X_1 \times X_2$ of codimension $d > 0$ and satisfying certain transversality conditions. Then morphisms (1.5) and (1.6) both have huge cones with non-zero cohomologies in all degrees from $-d$ to 0. However these two cones mostly annihilate each other and the cone of the composition (1.5)-(1.6) is actually quite small. Moreover, the calculation involved suggests an alternative decomposition of (1.5)-(1.6) which is better suited to computing cones - see (3.10) and (3.12).

The rest of Section 3 is devoted to showing that the underlying reasons are completely general and that a similar alternative decomposition of $Q \to \mathcal{O}_\Delta$ can be obtained for any $E$ which is a pushforward from a closed subscheme. Indeed, let $D \xrightarrow{\iota_D} X_1 \times X_2$ be a closed subscheme and let $E_D$ be a perfect object of $D(D)$ such that $E = \iota_{D*}E_D$ is also perfect. In Section 3.2 we show that, re-writing $E^\vee$ as a pushforward of an object from $D$, the evaluation map $E \otimes E^\vee \to \mathcal{O}_{X_1\times X_2}$ for $E$ on $X_1 \times X_2$ decomposes into

$$\iota_{D*}(-) \otimes \iota_{D*}(-) \to \iota_{D*}(- \otimes -) \tag{1.8}$$

$$\iota_{D*}\,(\text{The evaluation map for } E_D \text{ on } D) \tag{1.9}$$

$$\text{The adjunction counit } \iota_{D*}\iota_D^!(O_{X_1\times X_2}) \to O_{X_1\times X_2} \tag{1.10}$$

We substitute this instead of (1.6) into the decomposition (1.5)-(1.7) of $Q \to \mathcal{O}_\Delta$ and note that the right adjunction counits for $\iota_{D*}$ and for $\pi_{1*}$ compose to give the right adjunction counit for their composition $\pi_{D1*}$. We obtain:

$$\pi_{13*}\,(\text{Derived restriction to the diagonal } X_1 \times X_2 \text{ inside } X_1 \times X_2 \times X_1) \tag{1.11}$$

$$\Delta_*\pi_{1*}\,(\iota_{D*}(-) \otimes \iota_{D*}(-) \to \iota_{D*}(- \otimes -)) \tag{1.12}$$

$$\Delta_*\pi_{D1*}\,(\text{The evaluation map for } E_D \text{ on } D) \tag{1.13}$$

$$\Delta_*\left(\text{The adjunction counit } \pi_{D1*}\pi_{D1}^!(\mathcal{O}_{X_1}) \to \mathcal{O}_{X_1}\right) \tag{1.14}$$

The key observation is that the canonical map $\iota_{D*}(-) \otimes \iota_{D*}(-) \to \iota_{D*}(- \otimes -)$ should be viewed as the Künneth map (see Section 3.3 for the definition) for the fiber product square

$$\sigma: \quad \begin{tikzcd} D \arrow[r] \arrow[d] & D \arrow[d, "\iota_D"] \\ D \arrow[r, "\iota_D"'] & X_1 \times X_2 \end{tikzcd} \tag{1.15}$$

And this square is the restriction to the diagonal in $X_1 \times X_2 \times X_1$ of the fiber product square

$$\sigma': \quad \begin{tikzcd} D' \arrow[r] \arrow[d] & D_{23} \arrow[d, "\iota_{D23}"] \\ D_{12} \arrow[r, "\iota_{D12}"'] & X_1 \times X_2 \times X_1 \end{tikzcd} \tag{1.16}$$

where $D_{12} = \pi_{12}^{-1}D$, $D_{23} = \pi_{23}^{-1}D$, $D' = D_{12} \cap D_{23}$ and $\iota_\bullet$ are the corresponding inclusion maps. In Prps. 3.4 we prove a general statement which says that the Künneth map commutes with an arbitrary base change. In particular, this means that restricting to the diagonal $X_1 \times X_2$ inside $X_1 \times X_2 \times X_1$ and then doing the Künneth map for $\sigma$ is the same as first doing the Künneth map for $\sigma'$ and then restricting to the diagonal $D$ in $D'$. Hence we can re-write (1.11)-(1.14) as

$$\pi_{13*}\,(\text{The Künneth map for } \sigma') \tag{1.17}$$

$$\pi_{13*}\iota_{D'*}\,(\text{Derived restriction to the diagonal } D \text{ inside } D') \tag{1.18}$$

$$\Delta_*\pi_{D1*}\,(\text{The evaluation map for } E_D \text{ on } D) \tag{1.19}$$

$$\Delta_*\left(\text{The adjunction counit } \pi_{D1*}\pi_{D1}^!(\mathcal{O}_{X_1}) \to \mathcal{O}_{X_1}\right) \tag{1.20}$$

This is our preferred decomposition of the morphism $Q \to \mathcal{O}_\Delta$. Theorem 3.1 states it in full detail and Corollary 3.5 gives the parallel statement for the right adjunction counit. Note that for $D$ being the whole of $X_1 \times X_2$ the decomposition of Theorem 3.1 reduces simply to that of Theorem 2.1.

One advantage of the decomposition (1.17)-(1.20) is that it moves most of the action away from the ambient spaces $X_1 \times X_2 \times X_1$ and $X_1 \times X_2$ to their subschemes $D'$ and $D$. This allows us to get rid at last of the

assumption of $X_2$ being proper and replace it by the assumption that the support of $D$ is proper over $X_1$ and $X_2$ (see Theorem 3.1). Something that would no doubt be appreciated by those who want to do spherical twists on non-compact varieties such as the total spaces of the cotangent bundles of projective varieties.

Another advantage of the decomposition in Theorem 3.1 is that most of the morphisms in it can become isomorphisms under fairly reasonable assumptions on $E_D$ and $D$ - which makes computing the cone of $Q \to \mathcal{O}_\Delta$ much simpler a job. Indeed, observe that unlike the Künneth map for the square $\sigma$ which is never an isomorphism unless $D$ is the whole of $X_1 \times X_2$, the Künneth map for $\sigma'$ is an isomorphism whenever $D_{12}$ intersects $D_{23}$ transversally. The evaluation map for $E_D$ on $D$ in (1.19) is an isomorphism whenever $E_D$ is a line bundle on $D$ (or any other object such that $\mathcal{O}_D$ is reflexive with respect to it). And the adjunction counit (1.20) is an isomorphism whenever $D \xrightarrow{\pi_{D1}} X_1$ is such that $\pi_{D1*}\mathcal{O}_D = \mathcal{O}_{X_1}$, e.g. $D$ is a blowup of $X_1$ or a Fano fibration over it. This allows for a number of corollaries to Theorem 3.1 along the following lines:

**Corollary 1.2.** *Let $X_1$ and $X_2$ be a pair of separable schemes of finite type over a field $k$. Let $D \xrightarrow{\iota_D} X_1 \times X_2$ be a closed immersion proper over both $X_1$ and $X_2$. Denote by $\pi_{D1}$ the composition $D \xrightarrow{\iota_D} X_1 \times X_2 \xrightarrow{\pi_1} X_1$. Set $D_{12} = \pi_{12}^{-1}(D)$, $D_{23} = \pi_{23}^{-1}(D)$ and $D' = D_{12} \cap D_{23}$.*

*Then, if $D_{12}$ intersects $D_{23}$ transversally and if $\pi_{D1*}\mathcal{O}_D = \mathcal{O}_{X_1}$, the left twist of the Fourier-Mukai transform $\Phi_{\mathcal{O}_D}$ is the Fourier-Mukai transform with kernel $\pi_{13*}\iota_{D'*}\mathcal{I}_\Delta[1]$ where $\mathcal{I}_\Delta$ is the ideal sheaf on $D'$ of the diagonal $D \xrightarrow{\Delta} D'$.*

Finally, in Section 4 we give a concrete example of using Theorem 3.1 to compute the left twist of a Fourier-Mukai transform. We choose the transform to be the naive derived category transform induced by the Mukai flop. This transform is not equivalence - it was proved by Namikawa in [Nam03] by direct comparison of Hom spaces. It therefore has a non-trivial twist which can be computed quickly and efficiently by our method.

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## 2. The adjunction counit for Fourier-Mukai kernels

Given a scheme $X$ denote by $D_{\mathrm{qc}}(X)$ (resp. $D(X)$) the full subcategory of the derived category of $\mathcal{O}_X$-**Mod** consisting of complexes with quasi-coherent (resp. bounded and coherent) cohomology.

2.1. **Compact case.** Let $X_1$ and $X_2$ be a pair of separable schemes of finite type over a field $k$ and let $X_2$ be also proper. We have the following commutative diagram

(2.1)
$$\begin{tikzcd}
 & & X_1 \times X_2 \times X_1 \arrow[dl, "\pi_{12}"'] \arrow[d, "\pi_{13}"] \arrow[dr, "\pi_{23}"] & & \\
 & X_1 \times X_2 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & X_1 \times X_1 \arrow[dll, "\tilde{\pi}_1"] \arrow[drr, "\tilde{\pi}_2"'] & X_2 \times X_1 \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_1"] & \\
X_1 & & X_2 & & X_1
\end{tikzcd}$$

All the morphisms in it are separated, of finite-type and perfect ([Ill71], §4 or [Lip09], §4.9 for a less lofty approach) and, moreover, morphisms $\pi_1$ and $\pi_{13}$ are also proper.

Let $E$ be a perfect object of $D(X_1 \times X_2)$, i.e. it is quasi-isomorphic to a bounded complex of locally-free sheaves of finite rank (lffr). Let $\Phi_E \colon D_{\mathrm{qc}}(X_1) \to D_{\mathrm{qc}}(X_2)$ be the Fourier-Mukai transform with kernel $E$:

$$\Phi_E(-) = \pi_{2*}\left(E \otimes \pi_1^*(-)\right).$$

The left adjoint of $\pi_{2*} \colon D_{\mathrm{qc}}(X_1 \times X_2) \to D_{\mathrm{qc}}(X_2)$ is $\pi_2^* \colon D_{\mathrm{qc}}(X_2) \to D_{\mathrm{qc}}(X_1 \times X_2)$. This adjunction exists for an arbitrary morphism of ringed spaces on the level of the abelian categories of sheaves of modules ([GD60], Chap. 0, §4.4) and extends to an adjunction between their unbounded derived categories ([Lip09], Prop. 3.2.1).

The left and right adjoint of $(-)\otimes E$ as a functor from $D_{\rm qc}(X_1\times X_2)$ to itself is $(-)\otimes E^\vee$ where $E^\vee = \mathbf{R}\mathcal{H}om(E, \mathcal{O}_{X_1\times X_2})$. Very generally, the right adjoint of $(-)\otimes E$ is $\mathbf{R}\mathcal{H}om(E,-)$ ([Lip09], §2.6) and for a perfect $E$ we have $\mathbf{R}\mathcal{H}om(E,-)\simeq(-)\otimes E^\vee$.

The left adjoint of $\pi_1^*\colon D_{\rm qc}(X_2)\to D_{\rm qc}(X_1\times X_2)$ is obtained by the virtue of the following fact:

**Proposition 2.1** ([Lip09], Theorem 4.1.1, Prop. 4.7.1, [AIL10], Lemma 2.1.10)**.** *Let $f\colon X\to Y$ be a morphism of Noetherian schemes which is proper, perfect, separated and of finite type. Then:*

(1) *Functor $f_*\colon D_{qc}(X)\to D_{qc}(Y)$ has a right adjoint $f^!\colon D_{qc}(Y)\to D_{qc}(X)$.*
(2) *There is a natural isomorphism of functors $D_{qc}(X)\to D_{qc}(X)$:*
$$f^!(\mathcal{O}_Y)\otimes f^*(-)\xrightarrow{\sim} f^!(-).$$
(3) *The following morphism of functors $D_{qc}(Y)\to D_{qc}(X)$, adjoint to the isomorphism in (2), is itself an isomorphism:*
$$f^*(-)\xrightarrow{\sim}\mathbf{R}\mathcal{H}om\left(f^!(\mathcal{O}_Y), f^!(-)\right)$$

It follows that $\pi_1^*\colon D_{\rm qc}(X_1)\to D_{\rm qc}(X_1\times X_2)$ has a left adjoint which is isomorphic to $\pi_{1*}(-\otimes\pi_1^!(\mathcal{O}_{X_1}))$.

Therefore the left adjoint $\Phi_E^{\rm ladj}\colon D_{qc}(X_2)\to D_{qc}(X_1)$ is isomorphic to the Fourier-Mukai transform with kernel $E^\vee\otimes\pi_1^!(\mathcal{O}_{X_1})$:
$$\Phi_E^{\rm ladj}(-)\simeq\pi_{1*}\left(E^\vee\otimes\pi_1^!(\mathcal{O}_{X_1})\otimes\pi_2^*(-)\right).$$
By [Muk81], Prop. 1.3, the composition $\Phi_E^{\rm ladj}\Phi_E$ is isomorphic to the Fourer-Mukai transform $\Phi_Q\colon D_{\rm qc}(X_1)\to D_{\rm qc}(X_1)$ with kernel
$$Q=\pi_{13*}\left(\pi_{12}^*E\otimes\pi_{23}^*E^\vee\otimes\pi_{23}^*\pi_1^!(\mathcal{O}_{X_1})\right).$$

Let now $\Delta$ denote the diagonal inclusion $X_1\hookrightarrow X_1\times X_1$ and, by abuse of notation, let it also denote the induced inclusion $X_1\times X_2\hookrightarrow X_1\times X_2\times X_1$, so that there is the following commutative diagram:

(2.2)
$$\begin{tikzcd} X_1\times X_2 \arrow[r, hook, "\Delta"] \arrow[d, "\pi_1"'] & X_1\times X_2\times X_1 \arrow[d, "\pi_{13}"] \\ X_1 \arrow[r, hook, "\Delta"'] & X_1\times X_1 \end{tikzcd}$$

We can now state the main result of this section:

**Theorem 2.1.** *The adjunction counit $\gamma_E\colon\Phi_E^{ladj}\Phi_E\to\mathrm{Id}_{D_{qc}(X_1)}$ is isomorphic to the morphism of the Fourier-Mukai transforms $D_{qc}(X_1)\to D_{qc}(X_1)$ induced by the following morphism of objects of $D(X_1\times X_1)$:*

$$Q=\pi_{13*}\left(\pi_{12}^*E\otimes\pi_{23}^*E^\vee\otimes\pi_{23}^*\pi_1^!(\mathcal{O}_{X_1})\right)\to\pi_{13*}\Delta_*\Delta^*\left(\pi_{12}^*E\otimes\pi_{23}^*E^\vee\otimes\pi_{23}^*\pi_1^!(\mathcal{O}_{X_1})\right)\tag{2.3}$$
$$\pi_{13*}\Delta_*\Delta^*\left(\pi_{12}^*E\otimes\pi_{23}^*E^\vee\otimes\pi_{23}^*\pi_1^!(\mathcal{O}_{X_1})\right)\quad\simeq\quad\Delta_*\pi_{1*}\left(E\otimes E^\vee\otimes\pi_1^!(\mathcal{O}_{X_1})\right)\tag{2.4}$$
$$\Delta_*\pi_{1*}\left(E\otimes E^\vee\otimes\pi_1^!(\mathcal{O}_{X_1})\right)\to\Delta_*\pi_{1*}\left(\pi_1^!(\mathcal{O}_{X_1})\right)\tag{2.5}$$
$$\Delta_*\pi_{1*}\left(\pi_1^!(\mathcal{O}_{X_1})\right)\to\Delta_*\mathcal{O}_{X_1}.\tag{2.6}$$

*Here* (2.3) *is induced by the adjunction unit $\mathrm{Id}_{X_1\times X_2\times X_1}\to\Delta_*\Delta^*$,* (2.4) *is induced by the identities $\pi_{13}\circ\Delta=\Delta\circ\pi_1$ and $\pi_{12}\circ\Delta=\pi_{23}\circ\Delta=\mathrm{Id}_{X_1\times X_2}$ (see diagrams* (2.1)*-*(2.2)*),* (2.5) *is induced by the evaluation map $E\otimes E^\vee\to\mathcal{O}_{X_1\times X_2}$ and* (2.6) *- by the adjunction counit $\pi_{1*}\pi_1^!\to\mathrm{Id}_{X_1}$.*

To prove Theorem 2.1 we shall need the following two general lemmas, which state, essentially, that the projection formula commutes with the adjunction morphisms of a pushdown.

**Lemma 2.2.** *Let $S_1$, $S_2$, $S_3$ be ringed spaces and $S_1\xrightarrow{g}S_2\xrightarrow{f}S_3$ be ringed space morphisms. Let $A\in D(\mathcal{O}_{S_2}\text{-}\mathbf{Mod})$ and $B\in D(\mathcal{O}_{S_3}\text{-}\mathbf{Mod})$. Then the following diagram commutes:*

(2.7)
$$\begin{tikzcd} f_*A\otimes B \arrow[rr, "f_*\beta_g\otimes\mathrm{Id}_{S_3}"] \arrow[d, "\alpha_f"'] & & f_*g_*g^*A\otimes B \arrow[d, "\alpha_{fg}"] \\ f_*\left(A\otimes f^*B\right) \arrow[r, "f_*\beta_g"'] & f_*g_*g^*\left(A\otimes f^*B\right) \arrow[r, "\sim", "f_*g_*\nu_g"'] & f_*g_*\left(g^*A\otimes g^*f^*B\right). \end{tikzcd}$$

*Here $\alpha_f$ and $\alpha_{fg}$ are projection formula morphisms, $\beta_g$ is the adjunction morphism $\mathrm{Id}_{S_2} \to g_* g^*$ and $\nu_g$ is the bifunctorial isomorphism $g^*(- \otimes -) \xrightarrow{\sim} g^*(-) \otimes g^*(-)$ (see [Lip09], Prop. 3.2.4).*

*Proof of Lemma 2.2.* The idea is to show that the left adjoint of (2.7) with respect to $f_* g_*$ composed with the isomorphism

$$g^* f^* f_*(A) \otimes g^* f^*(B) \xrightarrow{\nu_{gf}^{-1}} g^* f^*(f_*(A) \otimes B)$$

commutes.

Recall that the projection formula morphism $\alpha_f$ is the right adjoint with respect to $f^*$ of

$$f^*(f_* A \otimes B) \xrightarrow{\nu_f} f^* f_* A \otimes f^* B \xrightarrow{\gamma_f \otimes \mathrm{Id}_{S_2}} A \otimes f^* B.$$

So taking the left adjoint with respect to $f_*$ of the lower-left half $f_* g_* \nu_g \circ f_* \beta_g \circ \alpha_f$ of (2.7) we obtain:

$$f^*(f_* A \otimes B) \xrightarrow{\nu_f} f^* f_* A \otimes f^* B \xrightarrow{\gamma_f \otimes \mathrm{Id}_{S_2}} A \otimes f^* B \xrightarrow{\beta_g} g_* g^*(A \otimes f^* B) \xrightarrow{g_* \nu_g} g_*(g^* A \otimes g^* f^* B)$$

Taking the left adjoint with respect to $g_*$ we obtain:

$$g^* f^*(f_* A \otimes B) \xrightarrow{g^* \nu_f} g^*(f^* f_* A \otimes f^* B) \xrightarrow{g^*(\gamma_f \otimes \mathrm{Id}_{S_2})} g^*(A \otimes f^* B) \xrightarrow{\nu_g} g^* A \otimes g^* f^* B$$

Composing this with $\nu_{gf}^{-1}$ on the left and noting that $g^* \nu_f \circ \nu_{gf}^{-1} = \nu_g^{-1}$ we obtain

$$g^* f^* f_* A \otimes g^* f^* B \xrightarrow{\nu_g^{-1}} g^*(f^* f_* A \otimes f^* B) \xrightarrow{g^*(\gamma_f \otimes \mathrm{Id}_{S_2})} g^*(A \otimes f^* B) \xrightarrow{\nu_g} g^* A \otimes g^* f^* B$$

which by the bifunctoriality of $\nu_g$ is just

$$g^* f^* f_*(A) \otimes g^* f^*(B) \xrightarrow{g^* \gamma_f \otimes \mathrm{Id}_{S_1}} g^* A \otimes g^* f^* B.$$

Treating similarly the upper-right half $\alpha_{fg} \circ f_* \beta_g \otimes \mathrm{Id}_{S_3}$ of (2.7) we obtain

$$g^* f^* f_*(A) \otimes g^* f^*(B) \xrightarrow{g^* f^* f_* \beta_g \otimes \mathrm{Id}_{S_1}} g^* f^* f_* g_* g^* A \otimes g^* f^* B \xrightarrow{\gamma_{gf} \otimes \mathrm{Id}_{S_1}} g^* A \otimes g^* f^* B$$

where $\gamma_{gf}$ is the adjunction morphism $g^* f^* f_* g_* \to \mathrm{Id}_{S_1}$.

Therefore the composition of the isomorphism $\nu_{gf}^{-1}$ with the left adjoint of (2.7) with respect to $f_* g_*$ is the tensor product of the triangle

$$\begin{tikzcd} g^* f^* f_*(A) \arrow[r, "g^* f^* f_* \beta_g"] \arrow[dr, "g^* \gamma_f"'] & g^* f^* f_* g_* g^* A \arrow[d, "\gamma_{gf}"] \\ & g^* A \end{tikzcd} \tag{2.8}$$

with the identity triangle on $g^* f^* B$. To see that (2.8) commutes take its right adjoint with respect to $g^* f^*$:

$$\begin{tikzcd} f_*(A) \arrow[r, "f_* \beta_g"] \arrow[dr, "f_* \beta_g"'] & f_* g_* g^* A \arrow[d, "\mathrm{Id}_{S_3}"] \\ & f_* g_* g^* A \end{tikzcd} \tag{2.9}$$

$\square$

**Lemma 2.3.** *Let $S_1$, $S_2$, $S_3$ be concentrated schemes and $S_1 \xrightarrow{g} S_2 \xrightarrow{f} S_3$ be quasi-perfect scheme maps. Let $A \in D_{qc}(S_2)$ and $B \in D_{qc}(S_3)$. Then the following diagram commutes:*

$$\begin{tikzcd} f_* g_* g^! A \otimes B \arrow[rr, "f_* \xi_g \otimes \mathrm{Id}_{S_3}"] \arrow[d, "\alpha_{fg}"'] & & f_* A \otimes B \arrow[d, "\alpha_f"] \\ f_* g_* \left(g^! A \otimes g^* f^* B\right) \arrow[r, "\sim", "f_* g_* \chi_g"'] & f_* g_* g^! \left(A \otimes f^* B\right) \arrow[r, "f_* \xi_g"'] & f_* \left(A \otimes f^* B\right). \end{tikzcd} \tag{2.10}$$

*Here $\alpha_i$ are projection formula isomorphisms, $\xi_g$ is the adjunction morphism $g_* g^! \to \mathrm{Id}_{S_2}$ and $\chi_g$ is the bifunctorial isomorphism $g^!(\text{-}) \otimes g^*(\text{-}) \xrightarrow{\sim} g^!(\text{-} \otimes \text{-})$ (see [Lip09], Exerc. 4.7.3.4).*

*Proof.* Similar to the proof of Lemma 2.2 we first compute the composition of the isomorphism $\nu_f^{-1}$ with the left adjoint of (2.10) with respect to $f_*$ to obtain:

(2.11)
$$\begin{tikzcd} f^* f_* g_* g^! A \otimes f^* B \arrow[rr, "f^* f_* \xi_g \otimes \mathrm{Id}_{S_2}"] \arrow[d, "\gamma_f \otimes \mathrm{Id}_{S_2}"'] & & f^* f_* A \otimes f^* B \arrow[dd, "\gamma_f \otimes \mathrm{Id}_{S_2}"] \\ g_* g^! A \otimes f^* B \arrow[d, "\alpha_g"] & & \\ g_* \left(g^! A \otimes g^* f^* B\right) \arrow[r, "\sim", "g_* \chi_g"'] & g_* g^! \left(A \otimes f^* B\right) \arrow[r, "\xi_g"'] & A \otimes f^* B. \end{tikzcd}$$

Observe that the right adjoint of $\xi_g \circ g_* \chi_g$ with respect to $g_*$ is $\chi_g$. On the other hand, $\chi_g$ is defined to be the right adjoint with respect to $g_*$ of $(\xi_g \otimes \mathrm{Id}_{S_2}) \circ \alpha_g^{-1}$. We conclude that $\xi_g \circ g_* \chi_g \circ \alpha_g = \xi_g \otimes \mathrm{Id}_{S_2}$ and so (2.11) is the tensor product of

(2.12)
$$\begin{tikzcd} f^* f_* g_* g^! A \arrow[r, "f^* f_* \xi_g"] \arrow[d, "\gamma_f"'] & f^* f_* A \arrow[d, "\gamma_f"] \\ g_* g^! A \arrow[r, "\xi_g"'] & A. \end{tikzcd}$$

with the identity square on $f^* B$. And (2.12) commutes by the functoriality of $\gamma_f$. $\square$

Back in the context of Theorem 2.1 we need the following crucial lemma, which explains where does the "restriction to diagonal" part of the morphism of kernels in Theorem 2.1 come from:

**Lemma 2.4.** *The following diagram of functors commutes:*

(2.13)
$$\begin{tikzcd} \pi_2^* \pi_{2*} \arrow[r, "\gamma_{\pi_2}"] \arrow[d, "\alpha"', "\simeq"] & \mathrm{Id}_{X_2 \times X_1} = \pi_{23*} \Delta_* \Delta^* \pi_{12}^* \\ \pi_{23*} \pi_{12}^* \arrow[ur, "\pi_{23*} \beta_\Delta"'] & \end{tikzcd}$$

*where $\alpha$ is the flat base change isomorphism ([Lip09], 3.9.5), $\gamma_{\pi_2}$ is the adjunction counit $\pi_2^* \pi_{2*} \to \mathrm{Id}_{X_1 \times X_2}$ and $\beta_\Delta$ is the adjunction unit $\mathrm{Id}_{X_1 \times X_2} \to \Delta_* \Delta^*$.*

*Proof.* To show that the composition

$$\pi_2^* \pi_{2*} \xrightarrow{\alpha} \pi_{23*} \pi_{12}^* \xrightarrow{\pi_{23*} \beta_\Delta} Id_{X_2 \times X_1}$$

is the adjunction morphism $\gamma_{\pi_2}$ it suffices to show that the morphism

$$\pi_{2*} \xrightarrow{\alpha'} \pi_{2*} \pi_{23*} \pi_{12}^* \xrightarrow{\pi_{2*} \pi_{23*} \beta_\Delta} \pi_{2*}, \tag{2.14}$$

where $\alpha'$ corresponds to $\alpha$ under the right adjunction on $\pi_2^*$, is the identity morphism. We have $\pi_2 \circ \pi_{23} = \pi_2 \circ \pi_{12}$ as scheme morphisms, so we can re-write (2.14) as

$$\pi_{2*} \xrightarrow{\alpha'} \pi_{2*} \pi_{12*} \pi_{12}^* \xrightarrow{\pi_{2*} \pi_{12*} \beta_\Delta} \pi_{2*} \tag{2.15}$$

The isomorphism $\alpha$ is constructed by applying $\pi_{2*}$ to the adjunction morphism $\beta_{\pi_{12}} \colon Id_{X_1 \times X_2} \to \pi_{12*} \pi_{12}^*$:

$$\pi_{2*} \xrightarrow{\pi_{2*} \beta_{\pi_{12}}} \pi_{2*} \pi_{12*} \pi_{12}^* = \pi_{2*} \pi_{23*} \pi_{12}^* \tag{2.16}$$

and then using the left adjunction on $\pi_{2*}$ to obtain $\alpha$. In other words, $\alpha'$ in (2.15) is precisely $\pi_{2*}\beta_{\pi_{12}}$ and, consequently, (2.15) is obtained by applying $\pi_{2*}$ to

$$\mathrm{Id}_{X_1\times X_2} \xrightarrow{\beta_{\pi_{12}}} \pi_{12*}\pi_{12}^* \xrightarrow{\pi_{12*}\beta_\Delta} \mathrm{Id}_{X_1\times X_2}\,. \tag{2.17}$$

The composition $\pi_{12*}\beta_\Delta \circ \beta_{\pi_{12}}$ is the adjunction morphism $\beta_{\pi_{12}\Delta}\colon \mathrm{Id}_{X_1\times X_2} \to \pi_{12*}\Delta_*\Delta^*\pi_{12}^*$ and it is the identity as morphism of functors since $\Delta\circ\pi_{12}$ is the identity as morphism of schemes. □

*Proof of Theorem 2.1.* Write $Q' = \pi_{23}^*\pi_1^!\mathcal{O}_{X_1}\otimes\pi_{23}^*E^\vee\otimes\pi_{12}^*E$ so that $Q = \pi_{13*}Q'$. Observe that as $\pi_{12}\circ\Delta = \pi_{23}\circ\Delta = \mathrm{Id}_{X_1\times X_2}$ we have $\Delta^*Q' = E^\vee\otimes E\otimes\pi_1^!\mathcal{O}_{X_1}$ and the evaluation map induces morphism $\Delta^*Q'\to\pi_1^!\mathcal{O}_{X_1}$. The Fourier-Mukai transform morphisms induced by kernel morphisms (2.3)-(2.6) are:

$$\tilde\pi_{2*}\left(\pi_{13*}Q'\otimes\tilde\pi_1^*(-)\right)\to\tilde\pi_{2*}\left(\pi_{13*}\Delta_*\Delta^*Q'\otimes\tilde\pi_1^*(-)\right) \tag{2.18}$$
$$\tilde\pi_{2*}\left(\pi_{13*}\Delta_*\Delta^*Q'\otimes\tilde\pi_1^*(-)\right)\xrightarrow{\sim}\tilde\pi_{2*}\left(\Delta_*\pi_{1*}\Delta^*Q'\otimes\tilde\pi_1^*(-)\right) \tag{2.19}$$
$$\tilde\pi_{2*}\left(\Delta_*\pi_{1*}\Delta^*Q'\otimes\tilde\pi_1^*(-)\right)\to\tilde\pi_{2*}\left(\Delta_*\pi_{1*}\pi_1^!\mathcal{O}_{X_1}\otimes\tilde\pi_1^*(-)\right) \tag{2.20}$$
$$\tilde\pi_{2*}\left(\Delta_*\pi_{1*}\pi_1^!\mathcal{O}_{X_1}\otimes\tilde\pi_1^*(-)\right)\to\tilde\pi_{2*}\left(\Delta_*\mathcal{O}_{X_1}\otimes\tilde\pi_1^*(-)\right) \tag{2.21}$$

On the other hand, the adjunction counit $\Phi_E^{\mathrm{ladj}}\Phi_E\to\mathrm{Id}_{D_{qc}(X_1)}$ is the composition of the following:

$$\pi_{1*}\left(\pi_1^!\mathcal{O}_{X_1}\otimes E^\vee\otimes\pi_2^*\pi_{2*}\left(E\otimes\pi_1^*(-)\right)\right)\to\pi_{1*}\left(\pi_1^!\mathcal{O}_{X_1}\otimes E^\vee\otimes E\otimes\pi_1^*(-)\right) \tag{2.22}$$
$$\pi_{1*}\left(\pi_1^!\mathcal{O}_{X_1}\otimes E^\vee\otimes E\otimes\pi_1^*(-)\right)\to\pi_{1*}\left(\pi_1^!\mathcal{O}_{X_1}\otimes\pi_1^*(-)\right) \tag{2.23}$$
$$\pi_{1*}\left(\pi_1^!\mathcal{O}_{X_1}\otimes\pi_1^*(-)\right)\to(-) \tag{2.24}$$

Here (2.22) is induced by the adjunction counit $\pi_2^*\pi_{2*}\to\mathrm{Id}_{X_1\times X_2}$, (2.23) is induced by the evaluation map $E\otimes E^\vee\to\mathcal{O}_{X_1\times X_2}$ and (2.24) is the adjunction counit for $\pi_1^*$ and its left adjoint $\pi_{1*}(\pi_1^!\mathcal{O}_{X_1}\otimes -)$.

Applying Lemma 2.4 we see that (2.22) is isomorphic to

$$\pi_{1*}\left(\pi_1^!\mathcal{O}_{X_1}\otimes E^\vee\otimes\pi_{23*}\pi_{12}^*\left(E\otimes\pi_1^*(-)\right)\right)\to\pi_{1*}\left(\pi_1^!\mathcal{O}_{X_1}\otimes E^\vee\otimes\pi_{23*}\Delta_*\Delta^*\pi_{12}^*\left(E\otimes\pi_1^*(-)\right)\right) \tag{2.25}$$

Now projection formula together with Lemma 2.2 imply that (2.25) is isomorphic to

$$\pi_{1*}\pi_{23*}\left(\pi_{23}^*\pi_1^!\mathcal{O}_{X_1}\otimes\pi_{23}^*E^\vee\otimes\pi_{12}^*E\otimes\pi_{12}^*\pi_1^*(-)\right)\to\pi_{1*}\pi_{23*}\Delta_*\Delta^*\left(\pi_{23}^*\pi_1^!\mathcal{O}_{X_1}\otimes\pi_{23}^*E^\vee\otimes\pi_{12}^*E\otimes\pi_{12}^*\pi_1^*(-)\right) \tag{2.26}$$

Using identities $\tilde\pi_1\circ\pi_{13}=\pi_1\circ\pi_{12}$ and $\tilde\pi_2\circ\pi_{13}=\pi_1\circ\pi_{23}$ (see diagram (2.1)) we can re-write (2.26) as

$$\tilde\pi_{2*}\pi_{13*}\left(Q'\otimes\pi_{13}^*\tilde\pi_1^*(-)\right)\to\tilde\pi_{2*}\pi_{13*}\Delta_*\left(\Delta^*Q'\otimes\Delta^*\pi_{13}^*\tilde\pi_1^*(-)\right) \tag{2.27}$$

Using further that $\pi_{13}\circ\Delta=\Delta\circ\pi_1$ (see diagram (2.2)), we write

$$\tilde\pi_{2*}\pi_{13*}\Delta_*\left(\Delta^*Q'\otimes\Delta^*\pi_{13}^*\tilde\pi_1^*(-)\right)\simeq\tilde\pi_{2*}\Delta_*\pi_{1*}\left(\Delta^*Q'\otimes\pi_1^*\Delta^*\tilde\pi_1^*(-)\right) \tag{2.28}$$

and then using $\Delta\circ\tilde\pi_1=\Delta\circ\tilde\pi_2=\mathrm{Id}_{X_1}$ we re-write (2.23) and (2.24) as

$$\tilde\pi_{2*}\Delta_*\pi_{1*}\left(\Delta^*Q'\otimes\pi_1^*\Delta^*\tilde\pi_1^*(-)\right)\to\tilde\pi_{2*}\Delta_*\pi_{1*}\left(\pi_1^!\mathcal{O}_{X_1}\otimes\pi_1^*\Delta^*\tilde\pi_1^*(-)\right) \tag{2.29}$$
$$\tilde\pi_{2*}\Delta_*\pi_{1*}\left(\pi_1^!\mathcal{O}_{X_1}\otimes\pi_1^*\Delta^*\tilde\pi_1^*(-)\right)\to\tilde\pi_{2*}\Delta_*\Delta^*\tilde\pi_1^*(-) \tag{2.30}$$

It follows from the above that the adjunction counit $\Phi_E^{\mathrm{ladj}}\Phi_E \to \mathrm{Id}_{D(X_1)}$ is isomorphic to the composition of (2.27)-(2.30). The claim of the theorem now follows from the fact that the following diagram commutes:

$$\begin{tikzcd}
\tilde{\pi}_{2*}\left(\pi_{13*}Q' \otimes \tilde{\pi}_1^*(-)\right) \arrow[r, "\sim"] \arrow[d, "(2.18)"'] & \tilde{\pi}_{2*}\pi_{13*}\left(Q' \otimes \pi_{13}^*\tilde{\pi}_1^*(-)\right) \arrow[d, "(2.27)"] \\
\tilde{\pi}_{2*}\left(\pi_{13*}\Delta_*\Delta^*Q' \otimes \tilde{\pi}_1^*(-)\right) \arrow[r, "\sim"] \arrow[d, "(2.19)"'] & \tilde{\pi}_{2*}\pi_{13*}\Delta_*\left(\Delta^*Q' \otimes \Delta^*\pi_{13}^*\tilde{\pi}_1^*(-)\right) \arrow[d, "(2.28)"] \\
\tilde{\pi}_{2*}\left(\Delta_*\pi_{1*}\Delta^*Q' \otimes \tilde{\pi}_1^*(-)\right) \arrow[r, "\sim"] \arrow[d, "(2.20)"'] & \tilde{\pi}_{2*}\Delta_*\pi_{1*}\left(\Delta^*Q' \otimes \pi_1^*\Delta^*\tilde{\pi}_1^*(-)\right) \arrow[d, "(2.29)"] \\
\tilde{\pi}_{2*}\left(\Delta_*\pi_{1*}\pi_1^!\mathcal{O}_{X_1} \otimes \tilde{\pi}_1^*(-)\right) \arrow[r, "\sim"] \arrow[d, "(2.21)"'] & \tilde{\pi}_{2*}\Delta_*\pi_{1*}\left(\pi_1^!\mathcal{O}_{X_1} \otimes \pi_1^*\Delta^*\tilde{\pi}_1^*(-)\right) \arrow[d, "(2.30)"] \\
\tilde{\pi}_{2*}\left(\Delta_*\mathcal{O}_{X_1} \otimes \tilde{\pi}_1^*(-)\right) \arrow[r, "\sim"] & \tilde{\pi}_{2*}\Delta_*\Delta^*\tilde{\pi}_1^*(-)
\end{tikzcd} \tag{2.31}$$

where the horizontal isomorphisms are all due to the projection formula. To see that diagram (2.31) indeed commutes, observe that its topmost square commutes by Lemma 2.2, the middle two commute trivially and the lowermost square commutes by Lemma 2.3. $\square$

An analogous statement for the right adjunction counit of a Fourier-Mukai transform differs from the one in Theorem 2.1 only by a slight change in the indices. Still, it is worth spelling out clearly. So let $X_1$, $X_2$ and $E$ be as above and let $\Psi_E$ be the Fourier-Mukai transform from $D(X_2)$ to $D(X_1)$ with kernel $E$. Similar to above, its right adjoint $\Psi_E^{\mathrm{radj}}$ is the Fourier-Mukai transform from $D(X_1)$ to $D(X_2)$ with kernel $E^\vee \otimes \pi_1^!(\mathcal{O}_{X_1})$. Therefore the composition $\Psi_E\Psi_E^{\mathrm{radj}}$ is the Fourier-Mukai transform $D(X_1) \to D(X_1)$ with the kernel

$$\tilde{Q} = \pi_{13*}\left(\pi_{12}^*E^\vee \otimes \pi_{23}^*E \otimes \pi_{12}^*\pi_1^!(\mathcal{O}_{X_1})\right) \tag{2.32}$$

Now an argument identical to the one in the proof of Theorem 2.1 yields the following:

**Corollary 2.5.** *The adjunction counit $\gamma_E\colon : \Psi_E\Psi_E^{radj} \to \mathrm{Id}_{D_{qc}(X_1)}$ is isomorphic to the morphism of Fourier-Mukai transforms $D_{qc}(X_1) \to D_{qx}(X_1)$ induced by the following morphism of objects of $D(X_1 \times X_1)$:*

$$\tilde{Q} = \pi_{13*}\left(\pi_{12}^*E^\vee \otimes \pi_{23}^*E \otimes \pi_{12}^*\pi_1^!(\mathcal{O}_{X_1})\right) \to \pi_{13*}\Delta_*\Delta^*\left(\pi_{12}^*E^\vee \otimes \pi_{23}^*E \otimes \pi_{12}^*\pi_1^!(\mathcal{O}_{X_1})\right) \tag{2.33}$$

$$\pi_{13*}\Delta_*\Delta^*\left(\pi_{12}^*E^\vee \otimes \pi_{23}^*E \otimes \pi_{12}^*\pi_1^!(\mathcal{O}_{X_1})\right) \simeq \Delta_*\pi_{1*}\left(E \otimes E^\vee \otimes \pi_1^!(\mathcal{O}_{X_1})\right) \tag{2.34}$$

$$\Delta_*\pi_{1*}\left(E \otimes E^\vee \otimes \pi_1^!(\mathcal{O}_{X_1})\right) \to \Delta_*\pi_{1*}\left(\pi_1^!(\mathcal{O}_{X_1})\right) \tag{2.35}$$

$$\Delta_*\pi_{1*}\left(\pi_1^!(\mathcal{O}_{X_1})\right) \to \Delta_*\mathcal{O}_{X_1}. \tag{2.36}$$

2.2. **Non-compact case.** In practice, one often has to deal with cases when neither $X_1$ nor $X_2$ are proper. A common way to deal with such situations is to restrict ourselves to the full subcategories of $D(X_1)$ and $D(X_2)$ consisting of objects with proper support. However, when the support of the kernel $E$ of the Fourier-Mukai transform is itself proper, it is still possible to work in full generality.

So let $X_1$ and $X_2$ be any two separable schemes of finite type over $\mathbb{C}$, not necessarily proper, and let $E$ be a perfect object of $D(X_1 \times X_2)$. If $\mathrm{Supp}(E)$ is proper over $X_2$ then the transform $\Phi_E$ still takes $D(X_1)$ to $D(X_2)$. If $\mathrm{Supp}(E)$ is proper over $X_1$, then, even though the left adjoint to $\pi_1^*$ no longer exists as a functor $D(X_1 \times X_2) \to D(X_1)$, the left adjoint to the composition $E \otimes \pi_1^*(-)$ does exist as a functor $D(X_1 \times X_2) \to D(X_1)$. The idea is that according to Deligne's work in [Del66] the left adjoint to $\pi_1^*$ exists naturally as a functor $\pi_{1!}$ from $D(X_1 \times X_2)$ to the category pro-$D(X_1)$, which is (roughly) the completion of $D(X_1)$ under taking inverse limits. It is then possible to show that on every object whose support is proper over $X_1$ this functor $\pi_{1!}$ is just the ordinary pushforward $\pi_{1*}$. This is described in detail in [Log08], Lemma 4. The result there is stated in terms of $X_2$ being smooth and so $\pi_1^!(\mathcal{O}_{X_1})$ being $\pi_2^*(\omega_{X_2})[\dim(X_2)]$, but exactly the same proof applies to give a following more general result:

**Proposition 2.6** ([Log08], Lemma 4)**.** *Let $X_1$ and $X_2$ be separable schemes of finite type over a field and let $E$ be a perfect object of $D(X_1 \times X_2)$ whose support is proper over $X_1$. Then the functor*

$$E \otimes \pi_1^*(-)\colon \quad D(X_1) \to D(X_1 \times X_2)$$

*has the left adjoint*

$$\pi_{1*}\left(E^\vee \otimes \pi_1^!(\mathcal{O}_{X_1}) \otimes (-)\right): \quad D(X_1 \times X_2) \to D(X_1)$$

Therefore the left adjoint $\Phi_E^{\mathrm{ladj}}$ to $\Phi_E$ still exists and we still have

$$\Phi_E^{\mathrm{ladj}}(-) = \pi_{1*}\left(E^\vee \otimes \pi_1^!(\mathcal{O}_{X_1}) \otimes \pi_2^*(-)\right)$$

and the composition $\Phi_E^{\mathrm{ladj}}\Phi_E$ is still the Fourier-Mukai transform $\Phi_Q : D(X_1) \to D(X_1)$ whose kernel is

$$Q = \pi_{13*}\left(\pi_{12}^* E \otimes \pi_{23}^* E^\vee \otimes \pi_{23}^*\pi_1^!(\mathcal{O}_{X_1})\right).$$

It therefore still makes sense to ask for a morphism of Fourier-Mukai kernels $Q \to \mathcal{O}_\Delta$ which induces the morphism of transforms isomorphic to the canonical adjunction morphism $\gamma_E\colon \Phi_E^{\mathrm{ladj}}\Phi_E \to \mathrm{Id}_{X_1}$.

To construct it, we first compactify $X_2$ - that is, we choose open immersion $\iota\colon X_2 \hookrightarrow \bar{X}_2$ with $\bar{X}_2$ proper. We shall abuse the notation by using $\iota$ to also denote immersions $X_1 \times X_2 \to X_1 \times \bar{X}_2$ and $X_1 \times X_2 \times X_1 \to X_1 \times \bar{X}_2 \times X_1$ where it causes no confusion. For any such compactified product space we shall denote by $\bar{\pi}_i$ and $\bar{\pi}_{ij}$ projections onto corresponding factors.

Denote by $\bar{E}$ the object $\iota_* E$ in $D(X_1 \times \bar{X}_2)$. Let $\Phi_{\bar{E}}$ and $\Phi_{\bar{E}}^{\mathrm{ladj}}$ be the corresponding Fourier-Mukai transform and its left-adjoint. The composition $\Phi_{\bar{E}}^{\mathrm{ladj}}\Phi_{\bar{E}}$ is the Fourier-Mukai transform $D(X_1) \to D(X_1)$ with the kernel

$$\bar{Q} = \bar{\pi}_{13*}\left(\bar{\pi}_{12}^* E \otimes \bar{\pi}_{23}^* E^\vee \otimes \bar{\pi}_{23}^*\bar{\pi}_1^!(\mathcal{O}_{X_1})\right).$$

**Proposition 2.7.** *Objects $\bar{Q}$ and $Q$ of $D(X_1 \times X_1)$ are isomorphic.*

*Proof.* We have a following chain of isomorphisms

$$\begin{aligned}(2.37)\qquad \bar{Q} &= \bar{\pi}_{13*}\left(\bar{\pi}_{12}^*\bar{E} \otimes \bar{\pi}_{23}^*\bar{E} \otimes \bar{\pi}_{23}^*\bar{\pi}_1^!(\mathcal{O}_{X_1})\right) = \bar{\pi}_{13*}\left(\bar{\pi}_{12}^*\iota_* E \otimes \bar{\pi}_{23}^*\iota_* E \otimes \bar{\pi}_{23}^*\bar{\pi}_1^!(\mathcal{O}_{X_1})\right) \xrightarrow{\sim} \\ &\xrightarrow{\sim} \bar{\pi}_{13*}\left(\iota_*\pi_{12}^* E \otimes \iota_*\pi_{23}^* E \otimes \bar{\pi}_{23}^*\bar{\pi}_1^!(\mathcal{O}_{X_1})\right) \xrightarrow{\sim} \bar{\pi}_{13*}\iota_*\left(\pi_{12}^* E \otimes \iota^*\iota_*\pi_{23}^* E \otimes \iota^*\bar{\pi}_{23}^*\bar{\pi}_1^!(\mathcal{O}_{X_1})\right) \xrightarrow{\sim} \\ &\xrightarrow{\sim} \bar{\pi}_{13*}\iota_*\left(\pi_{12}^* E \otimes \pi_{23}^* E \otimes \iota^*\bar{\pi}_{23}^*\bar{\pi}_1^!(\mathcal{O}_{X_1})\right) \xrightarrow{\sim} \pi_{13*}\left(\pi_{12}^* E \otimes \pi_{23}^* E \otimes \pi_{23}^*\pi_1^!(\mathcal{O}_{X_1})\right) = Q\end{aligned}$$

where the first isomorphism is due to the flat base change, the second due to the projection formula, the third due to the adjunction counit $\gamma_\iota\colon \iota^*\iota_* \to \mathrm{Id}$ being an isomorphism for any open immersion $\iota$ ([GD60], *Prop. 9.4.2*) and the last one due to the identities $\bar{\pi}_{13} \circ \iota = \pi_{13}$ and $\bar{\pi}_{23} \circ \iota = \pi_{23} \circ \iota$ and the fact that $\iota^*\bar{\pi}_1^! \simeq \pi_1^!$ as $\iota^!$ is defined to be $\iota^*$ for $\iota$ an open immersion (see [Del66]). $\square$

Therefore the functors $\Phi_{\bar{E}}^{\mathrm{ladj}}\Phi_{\bar{E}}$ and $\Phi_E^{\mathrm{ladj}}\Phi_E$ are isomorphic. This doesn't in itself mean that the adjunction counits $\gamma_{\bar{E}}\colon \Phi_{\bar{E}}^{\mathrm{ladj}}\Phi_{\bar{E}} \to \mathrm{Id}_{D(X_1)}$ and $\gamma_E\colon \Phi_E^{\mathrm{ladj}}\Phi_E \to \mathrm{Id}_{D(X_1)}$ also have to be isomorphic, however it does turn out to be the case.

Indeed, we have a following commutative diagram:

$$\begin{tikzcd} & X_1 \times \bar{X}_2 \arrow[dl, "\bar{\pi}_1"'] \arrow[dr, "\bar{\pi}_2"] & & X_1 \times X_2 \arrow[ll, hook', "\iota"'] \arrow[dr, "\pi_2"] & \\ X_1 & & \bar{X}_2 & & X_2 \arrow[ll, hook', "\iota"] \end{tikzcd} \tag{2.38}$$

**Lemma 2.8.** *There is an isomorphism of functors $D(X_1) \to D^b_{qc}(\bar{X}_2)$*

$$\alpha\colon \Phi_{\bar{E}} \xrightarrow{\sim} \iota_*\Phi_E \tag{2.39}$$

*Its left adjoint with respect to $\iota_*$ is an isomorphism of functors $D(X_1) \to D(X_2)$:*

$$\alpha'\colon \iota^*\Phi_{\bar{E}} \xrightarrow{\sim} \Phi_E \tag{2.40}$$

*Proof.* For the first claim, we set $\alpha$ to be

$$\Phi_{\bar{E}} = \bar{\pi}_{2*}\left(\bar{E} \otimes \bar{\pi}_1^*(-)\right) = \bar{\pi}_{2*}\left(\iota_* E \otimes \bar{\pi}_1^*(-)\right) \xrightarrow{\sim}$$
$$\xrightarrow{\sim} \bar{\pi}_{2*}\iota_*\left(E \otimes \iota^*\bar{\pi}_1^*(-)\right) \xrightarrow{\sim} \iota^*\pi_{2*}\left(E \otimes \pi_1^*(-)\right) = \Phi_E$$

where the first isomorphism is by the projection formula and the second by the commutativity of (2.38).

For the second claim: $\alpha'$ is the composition of $\iota^*\alpha$ with the adjunction counit $\gamma_\iota \colon \iota^*\iota_*\Phi_E \to \Phi_E$. And $\gamma_\iota$ is an isomorphism since $\iota$ is an open immersion ([GD60], *Prop. 9.4.2*). $\square$

The isomorphism $\alpha$ induces an isomorphism

$$\alpha'' \colon \Phi^{ladj}_{\bar{E}}\Phi_{\bar{E}} \xrightarrow{\sim} \Phi^{ladj}_E \iota^*\iota_*\Phi_E$$

which fits into the following commutative diagram of functors $D(X_1) \to D(X_1)$

(2.41)
$$\begin{tikzcd}
\Phi^{ladj}_{\bar{E}}\Phi_{\bar{E}} \arrow[d, "\alpha''"', "\sim"] \arrow[drr, "\gamma_{\bar{E}}"] & & \\
\Phi^{ladj}_E \iota^*\iota_*\Phi_E \arrow[r, "\Phi^{ladj}_E\gamma_\iota"'] & \Phi^{ladj}_E\Phi_E \arrow[r, "\gamma_E"'] & \mathrm{Id}_{D(X_1)}
\end{tikzcd}$$

Since $\gamma_\iota$ is an isomorphism we obtain

**Corollary 2.9.** *The adjunction counits $\gamma_{\bar{E}} \colon \Phi^{ladj}_{\bar{E}}\Phi_{\bar{E}} \to \mathrm{Id}_{D(X_1)}$ and $\gamma_E \colon \Phi^{ladj}_E\Phi_E \to \mathrm{Id}_{D(X_1)}$ are naturally isomorphic.*

We can therefore reduce constructing $\gamma_E$ on the level of Fourier-Mukai kernels to the construction of $\gamma_{\bar{E}}$:

**Proposition 2.10.** *Let $Q \to \mathcal{O}_\Delta$ be a composition of any isomorphism $Q \xrightarrow{\sim} \bar{Q}$ (e.g. that in (2.37)) with the morphism $\bar{Q} \to \mathcal{O}_\Delta$ given in Theorem 2.1. Then the induced morphism $\Phi_Q \to \Phi_{\mathcal{O}_\Delta}$ of Fourier-Mukai transforms is isomorphic to the adjunction counit $\Phi^{ladj}_E\Phi_E \to \mathrm{Id}_{D(X_1)}$*

## 3. An alternative description for the pushforward kernels

Whenever $E$ is a pushforward of an object from some subscheme of $X_1 \times X_2$ the decomposition of Theorem 2.1 can be far from a convenient way to compute the twist of $\Phi_E$. We first illustrate this in Section 3.1 with an example where $E$ is the structure sheaf of a global complete intersection subscheme and so everything can be worked out explicitly using Koszul-type resolutions. For a general closed subscheme of $X_1 \times X_2$ such a resolution doesn't exist and a different approach is needed. But with an insight obtained from Section 3.1 we set up some general machinery in Sections 3.2 and 3.3 which we then apply in Section 3.4 to obtain a better description of the morphism $Q \to \mathcal{O}_\Delta$ for $E$ being a pushforward from an arbitrary closed subscheme.

3.1. **The global complete intersection example.** Let $X_1$ and $X_2$ be a pair of separable schemes of finite type over a field $k$ and let $X_2$ be proper. Let $D \overset{\iota_D}{\hookrightarrow} X_1 \times X_2$ be a subscheme of codimension $d$ which is a complete intersection of divisors $D_1 \cap \dots \cap D_d$ and assume that $D_{12} = \pi_{12}^{-1}(D)$ intersects $D_{23} = \pi_{23}^{-1}(D)$ tranversally in $X_1 \times X_2 \times X_1$. We would like to compute the twist of $\Phi_{\mathcal{O}_D}$.

The structure sheaf $\mathcal{O}_D$ has a global Koszul-type resolution by lffr sheaves in $D(X_1 \times X_1)$:

(3.1)
$$\wedge^d \mathcal{N}^\vee \to \wedge^{d-1}\mathcal{N}^\vee \to \dots \to \mathcal{N}^\vee \to \mathcal{O}_{X_1 \times X_2}$$

where $\mathcal{N} = \bigoplus_i \mathcal{O}(D_i)$. The differential maps $\wedge^k \mathcal{N}^\vee \to \wedge^{k-1}\mathcal{N}^\vee$ are defined by

$$s_{i_1} \wedge \dots \wedge s_{i_k} \mapsto \sum_{j=1}^{k} idl_{i_j}(s_{i_j}) \quad \dots \wedge \widehat{s_{i_j}} \wedge \dots \qquad s_{i_k} \in \mathcal{O}(-D_{i_k})$$

where $idl_{i_j}$ is the inclusion $\mathcal{O}(-D_{i_j}) \hookrightarrow \mathcal{O}_{X_1 \times X_2}$. It is clear that all these maps vanish along $D$.

The object $\mathcal{O}_D \otimes \mathcal{O}_D^\vee$ is therefore given by the restriction of the dual of the complex (3.1) to $D$. Since all the differentials vanish along $D$ we obtain

(3.2)
$$\mathcal{O}_D \xrightarrow{0} \mathcal{N}|_D \xrightarrow{0} \dots \xrightarrow{0} \wedge^d \mathcal{N}|_D$$

that is

$$\bigoplus_{i=0}^{d} \wedge^i \mathcal{N}|_D[i] \tag{3.3}$$

The evaluation map $\mathcal{O}_D \otimes \mathcal{O}_D^\vee \to \mathcal{O}_{X_1 \times X_2}$ decomposes as the direct sum of maps $\wedge^i \mathcal{N}|_D[i] \to \mathcal{O}_{X_1\times X_2}$ given by maps of complexes

$$\begin{tikzcd} \overset{\text{deg. } i-d}{\wedge^d \mathcal{N}^\vee \otimes \wedge^i \mathcal{N}} \arrow[r] & \cdots \arrow[r] & \overset{\text{deg. } 0}{\wedge^i \mathcal{N}^\vee \otimes \wedge^i \mathcal{N}} \arrow[r] \arrow[d, "\text{evaluation map}"] & \cdots \arrow[r] & \overset{\text{deg. } i}{\mathcal{O}_{X_1\times X_2} \otimes \wedge^i \mathcal{N}} \\ & & \mathcal{O}_{X_1\times X_2} & & \end{tikzcd} \tag{3.4}$$

where we've marked explicitly the degrees of the elements of the complex at the top. Observe now that the complex dual to the complex at the top of (3.4) is exact in all degrees except for $d-i$. The dual of a complex being exact in some degree $k$ is equivalent to all maps from this complex to $\mathcal{O}_{X_1\times X_2}[k]$ being homotopic to zero. Therefore any map from the complex at the top of (3.4) to $\mathcal{O}_{X_1\times X_2}[0]$ is homotopic to zero unless $i = d$. In particular, all maps (3.4) are homotopic to zero except for the one where $i = d$, and that map simplifies to

$$\begin{tikzcd} \mathcal{O}_{X_1\times X_2} \arrow[r] \arrow[d, "\sim"'] & \cdots \arrow[r] & \wedge^i \mathcal{N} \arrow[r] & \cdots \arrow[r] & \wedge^d \mathcal{N} \\ \mathcal{O}_{X_1\times X_2} & & & & \end{tikzcd} \tag{3.5}$$

which is clearly the map $\mathcal{O}_D^\vee \to \mathcal{O}_{X_1\times X_2}$ dual to the restriction map $\mathcal{O}_{X_1\times X_2} \to \mathcal{O}_D$.

We conclude that the evaluation map decomposes as

$$\bigoplus_{i=0}^{d} \wedge^i \mathcal{N}|_D[i] \to \wedge^d \mathcal{N}|_D[d] \simeq \mathcal{O}_D^\vee \to \mathcal{O}_{X_1\times X_2}. \tag{3.6}$$

On the other hand, the object $\pi_{12}^* \mathcal{O}_D \otimes \pi_{23}^* \mathcal{O}_D^\vee$ is given in $D(X_1 \times X_2 \times X_1)$ by the complex

$$\overset{\text{deg. } 0}{\mathcal{O}_{D_{12}}} \to \mathcal{N}_{23}|_{D_{12}} \to \cdots \to \overset{\text{deg. } d}{\wedge^d \mathcal{N}_{23}|_{D_{12}}} \tag{3.7}$$

where $\mathcal{N}_{23} = \pi_{23}^* \mathcal{N}$. By the assumption that $D_{12}$ and $D_{23}$ intersect transversally this complex is exact everywhere except in degree $d$. And in degree $d$ its cohomology is just $\wedge^d \mathcal{N}_{23}|_{D'}$ where $D' = D_{12} \cap D_{23}$. The adjunction counit $\beta_\Delta\colon \mathrm{Id} \to \Delta_* \Delta^*$ on $X_1 \times X_2 \times X_1$ is simply the derived restriction to the diagonal $\Delta(X_1 \times X_2)$. Since $\Delta(X_1 \times X_2)$ intersects $D_{12}$ transversally, the derived restriction to $\Delta(X_1 \times X_2)$ of the pushforward of an object from $D_{12}$ is the pushforward from $D_{12}$ of the derived restriction to $\Delta D = D_{12} \cap \Delta(X_1 \times X_2)$. Therefore

$$\pi_{12}^* \mathcal{O}_D \otimes \pi_{23}^* \mathcal{O}_D^\vee \xrightarrow{\beta_\Delta} \Delta_* \Delta^* \pi_{12}^* \mathcal{O}_D \otimes \pi_{23}^* \mathcal{O}_D^\vee \simeq \Delta_* \left( \mathcal{O}_D \otimes \mathcal{O}_D^\vee \right) \tag{3.8}$$

is given by the map of complexes $D_{12} \to \Delta D$:

$$\begin{tikzcd} \mathcal{O}_{D_{12}} \arrow[r] \arrow[d] & \mathcal{N}_{23}|_{D_{12}} \arrow[r] \arrow[d] & \cdots \arrow[r] & \wedge^d \mathcal{N}_{23}|_{D_{12}} \arrow[d] \\ \mathcal{O}_{\Delta D} \arrow[r, "0"] & \mathcal{N}|_{\Delta D} \arrow[r, "0"] & \cdots \arrow[r, "0"] & \wedge^d \mathcal{N}|_{\Delta D} \end{tikzcd} \tag{3.9}$$

where each vertical arrow is sheaf restriction from $D_{12}$ to $\Delta D$. Hence the composition of (3.9) with the projection to $\wedge^d \mathcal{N}|_{\Delta D}$ is equivalent in $D(X_1 \times X_2 \times X_1)$ to the sheaf restriction $\wedge^d \mathcal{N}_{23}|_{D'}[d] \to \wedge^d \mathcal{N}|_{\Delta D}[d]$. We conclude that the composition of (3.9) with the evaluation map (3.6) is

$$\wedge^d \mathcal{N}_{23}|_{D'}[d] \to \wedge^d \mathcal{N}|_{\Delta D}[d] \simeq \Delta_* \mathcal{O}_D^\vee \to \Delta_* \mathcal{O}_{X_1\times X_2} \tag{3.10}$$

We now see that in our case the decomposition

$$\pi_{13*}(\pi_{12}^*\mathcal{O}_D \otimes \pi_{23}^*\mathcal{O}_D^\vee \otimes \pi_{23}^*\pi_1^!\mathcal{O}_{X_1}) \to \pi_{13*}\Delta_*\left(\mathcal{O}_D \otimes \mathcal{O}_D^\vee \otimes \pi_1^!\mathcal{O}_{X_1}\right) \to \pi_{13*}\Delta_*\pi_1^!\mathcal{O}_{X_1} \tag{3.11}$$

of Theorem 2.1 is not very practical from the point of view of computing cones. It goes (at least before applying $\pi_{13*}$) from a sheaf concentrated in degree $d$ to a huge complex with non-zero cohomologies in all degrees from 0 to $d$ and then to a sheaf concentrated in degree 0. We get two huge cones with non-zero cohomologies in all degrees from 0 to $d$ which then almost entirely annihilate each other when we take the cone of the map between them.

As evidenced by the above, a less wasteful approach is to break up the evaluation map into $\mathcal{O}_D \otimes \mathcal{O}_D^\vee \to \mathcal{O}_D^\vee$ followed by $\mathcal{O}_D^\vee \to \mathcal{O}_{X_1\times X_2}$ and then re-group (3.8) as

$$\pi_{13*}(\pi_{12}^*\mathcal{O}_D \otimes \pi_{23}^*\mathcal{O}_D^\vee \otimes \pi_{23}^*\pi_1^!\mathcal{O}_{X_1}) \to \pi_{13*}\Delta_*\left(\mathcal{O}_D^\vee \otimes \pi_1^!\mathcal{O}_{X_1}\right) \to \pi_{13*}\Delta_*\pi_1^!\mathcal{O}_{X_1} \tag{3.12}$$

These are precisely the two morphisms in (3.10): the sheaf restriction of $\wedge^d\mathcal{N}_{23}$ from $D'$ to $\Delta D$ in degree $d$, followed by the dual of the restriction $\mathcal{O}_{X_1\times X_2} \to \mathcal{O}_D$ in degree 0. Both of these cones are small compared to those in (3.11) and easy to compute.

It turns out that this approach can be made to work in a much more general setting than that of a global complete intersection subscheme.

3.2. **A decomposition of the evaluation map.** Let $X$ and $Y$ be a pair of concentrated schemes and $f\colon Y \to X$ be a quasi-proper, finitely presentable scheme map. Let $E$ be an object of $D_{qc}(Y)$. By the sheafifed Grothendieck duality ([Lip09], §4.4) we have an isomorphism

$$\delta_f\colon f_*\,\mathbf{R}\mathcal{H}om\left(E, f^!(\mathcal{O}_X)\right) \xrightarrow{\sim} \mathbf{R}\mathcal{H}om\left(f_*E, \mathcal{O}_X\right). \tag{3.13}$$

**Proposition 3.1.** *The following diagram commutes*

$$\begin{tikzcd} f_*E \otimes \mathbf{R}\mathcal{H}om(f_*E,\mathcal{O}_X) \arrow[ddd, "ev(\iota_*E)"'] & f_*E \otimes f_*\mathbf{R}\mathcal{H}om(E, f^!\mathcal{O}_X) \arrow[l, "\mathrm{Id}_X\otimes\delta_f"', "\sim"] \arrow[d, "\kappa_f"] \\ & f_*\left(E \otimes \mathbf{R}\mathcal{H}om(E, f^!\mathcal{O}_X)\right) \arrow[d, "f_*ev(E)"] \\ & f_*f^!\mathcal{O}_X \arrow[d, "\xi_f"] \\ & \mathcal{O}_X \end{tikzcd} \tag{3.14}$$

*Here $\kappa_f$ is the canonical morphism $f_*(-)\otimes f_*(-) \to f_*(-)$, $\xi_f$ is the adjunction morphism $f_*f^! \to \mathrm{Id}_X$ and* $\mathrm{ev}(E)$ *is the evaluation map $E \otimes \mathbf{R}\mathcal{H}om(E,-) \to (-)$.*

*Proof.* The canonical morphism $\kappa_f$ can be decomposed as

$$f_*(-)\otimes f_*(-) \xrightarrow{\alpha_f} f_*\left(-\otimes f^*f_*(-)\right) \xrightarrow{f_*(\mathrm{Id}\otimes\gamma_f)} f_*(-\otimes -)$$

where $\alpha_f$ is the projection formula isomorphism and $\gamma_f$ is the adjunction morphism $f^* f_* \to \mathrm{Id}_Y$. It therefore suffice to show that the diagram

$$\begin{tikzcd} f_* E \otimes \mathbf{R}\mathcal{H}om\left(f_* E, \mathcal{O}_X\right) \arrow[ddddrr, "ev(\iota_* E)"'] & & f_*\left(E \otimes f^* f_* \mathbf{R}\mathcal{H}om\left(E, f^! \mathcal{O}_X\right)\right) \arrow[ll, "\alpha_f^{-1}\circ(\mathrm{Id}_X \otimes \delta_f)"', "\sim"] \arrow[d, "f_*(E \otimes \gamma_f)"] \\ & & f_*\left(E \otimes \mathbf{R}\mathcal{H}om\left(E, f^! \mathcal{O}_X\right)\right) \arrow[d, "f_* ev(E)"] \\ & & f_* f^! \mathcal{O}_X \arrow[d, "\xi_f"] \\ & & \mathcal{O}_X \end{tikzcd} \tag{3.15}$$

commutes. But observe that the projection formula isomorphism $\alpha_f$ induces an isomorphism between the functors $f_* E \otimes (-)$ and $f_*\left(E \otimes f^*(-)\right)$, while the sheafified Grothendieck duality isomorphism $\delta_f$ induces an isomorphism between their right adjoints $\mathbf{R}\mathcal{H}om(f_* E, -)$ and $f_* \mathbf{R}\mathcal{H}om\left(E, f^!(-)\right)$. Moreover, the map $\mathrm{ev}(f_* E)$ is the adjunction counit for the adjoint pair $(f_* E \otimes (-), \mathbf{R}\mathcal{H}om(f_* E, -))$, while the three morphisms in the right column of (3.15) are precisely the adjunction counits for $(f^*, f_*)$, $(E \otimes (-), \mathbf{R}\mathcal{H}om(E, -))$ and $\left(f_*, f^!\right)$, so their composition is the adjunction counit for the pair $\left(f_*\left(E \otimes f^*(-)\right), f_*\left(\mathbf{R}\mathcal{H}om(E, f^!(-))\right)\right)$. We conclude that (3.15) commutes by the uniqueness of the adjoint pairs and their units/counits. $\square$

The reader can now check that if we set the morphism $f\colon Y \to X$ in Proposition 3.1 to be the inclusion $\iota_D\colon D \hookrightarrow X_1 \times X_2$ of Section 3.1 and set $E = \mathcal{O}_D$, then the decomposition of the evaluation map in Proposition 3.1 is precisely the decomposition (3.6) of Section 3.1. First we identify $(\iota_{D*}\mathcal{O}_D)^\vee$ with $\iota_{D*}\iota_D^!\mathcal{O}_{X_1\times X_2}$ via the Grothendieck duality. The canonical morphism $\kappa_D\colon\ \iota_{D*}\mathcal{O}_D \otimes \iota_{D*}\iota_D^!\mathcal{O}_{X_1\times X_2} \to \iota_{D*}\iota_D^!\mathcal{O}_{X_1\times X_2}$ is precisely the projection $\bigoplus_{i=0}^d \wedge^i \mathcal{N}|_D[i] \to \wedge^d \mathcal{N}|_D[d]$. The evaluation map for $\mathcal{O}_D$ on $D$ is trivial, and finally the adjunction morphism $\gamma_D\colon \iota_{D*}\iota_D^!\mathcal{O}_{X_1\times X_2} \to \mathcal{O}_{X_1\times X_2}$ is precisely the map dual to the restriction $\mathcal{O}_{X_1\times X_2} \to \mathcal{O}_D$ under the identification of $(\iota_{D*}\mathcal{O}_D)^\vee$ and $\iota_{D*}\iota_D^!\mathcal{O}_{X_1\times X_2}$ via the Grothendieck duality.

3.3. **Künneth maps and the base change.** Let $X$, $Y$ and $f\colon Y \to X$ be as in Section 3.2. The canonical morphism $\kappa_f\colon f_*(-) \otimes f_*(-) \to f_*(- \otimes -)$ can be interpreted as the Künneth map of the commutative square:

$$\begin{tikzcd} Y \arrow[r, "\mathrm{Id}"] \arrow[d, "\mathrm{Id}"'] & Y \arrow[d, "f"] \\ Y \arrow[r, "f"'] & X \end{tikzcd} \tag{3.16}$$

We recall the basics on the Künneth map (see [Lip09] for more detail):

**Definition 3.2.** Let

$$\begin{tikzcd} Z \arrow[r, "g_2"] \arrow[d, "g_1"'] & Y_2 \arrow[d, "f_2"] \\ Y_1 \arrow[r, "f_1"'] & X \end{tikzcd} \tag{3.17}$$

be a commutative square $\sigma$ of concentrated schemes. Setting $h = f_1 \circ g_1 = f_2 \circ g_2$ define the *Künneth map* to be the bifunctorial morphism

$$\kappa_\sigma\colon\ f_{1*}(A_1) \otimes f_{2*}(A_2) \to h_*\left(g_1^*(A_1) \otimes g_2^*(A_2)\right) \qquad A_i \in D(Y_i) \tag{3.18}$$

which is the composition

$$f_{1*}(A_1) \otimes f_{2*}(A_2) \xrightarrow{\beta_h} h_* h^* \left(f_{1*}(A_1) \otimes f_{2*}(A_2)\right) \simeq$$
$$\simeq h_* \left(g_1^* f_1^* f_{1*}(A_1) \otimes g_2^* f_2^* f_{2*}(A_2)\right) \xrightarrow{h_*(\gamma_{f_1} \otimes \gamma_{f_2})} h_* \left(g_1^*(A_1) \otimes g_2^*(A_2)\right) \tag{3.19}$$

with $\beta_h$ being the adjunction morphism $\mathrm{Id}_X \to h_* h^*$ and $\gamma_{f_i}$ being adjunction morphisms $f_i^* f_{i*} \to \mathrm{Id}_{Y_i}$.

A commutative square is called *Künneth-independent*when its Künneth map is an isomorphism. For fiber squares of concentrated schemes this notion of independence is equivalent to several others:

**Proposition 3.3** ([Lip09], Theorem 3.10.3)**.** *For a fiber square of concentrated schemes*

$$\begin{tikzcd} Z = Y_1 \times_X Y_2 \arrow[r, "g_2"] \arrow[d, "g_1"'] & Y_2 \arrow[d, "f_2"] \\ Y_1 \arrow[r, "f_1"'] & X \end{tikzcd} \tag{3.20}$$

*the following are equivalent:*

1. *The square is independent, i.e. the base change map $f_1^* f_{2*} \to g_2^* g_{1*}$ is a functorial isomorphism.*
2. *The square is Künneth-independent.*
3. *The square is* Tor*-independent, i.e. for any pair of points $y_1 \in Y_1$ and $y_2 \in Y_2$ such that $f_1(y_1) = f_2(y_2) = x \in X$ we have*

$$\mathrm{Tor}^i_{\mathcal{O}_{X,x}}(\mathcal{O}_{Y_1,y_1}, \mathcal{O}_{Y_2,y_2}) = 0 \textit{ for all } i > 0. \tag{3.21}$$

In these terms one can now roughly describe what happened in Section 3.1 as follows: restricting to diagonal $\Delta$ in $X_1 \times X_2 \times X_1$ followed by the Künneth map for the square

$$\begin{tikzcd} D \arrow[r] \arrow[d] & D \arrow[d] \\ D \arrow[r] & X_1 \times X_2 \end{tikzcd} \tag{3.22}$$

turned out to be the same as first doing the Künneth map for

$$\begin{tikzcd} D' = D_{12} \cap D_{23} \arrow[r] \arrow[d] & D_{23} \arrow[d] \\ D_{12} \arrow[r] & X_1 \times X_2 \times X_1 \end{tikzcd} \tag{3.23}$$

and then restricting to diagonal $\Delta D$ in $D'$. And as $D_{12}$ intersects $D_{23}$ transversally the fiber square (3.23) is Tor-independent, and therefore Künneth-independent. Hence the Künneth map for (3.23) is an isomorphism, so to compute cones one only needs to take the cone of the restriction to $\Delta D$ in $D'$.

This turns out to be a special case of a very general base change statement for Künneth maps:

**Proposition 3.4.** *Let*

$$\begin{tikzcd} Z \arrow[r, "g_2"] \arrow[d, "g_1"'] & Y_2 \arrow[d, "f_2"] \\ Y_1 \arrow[r, "f_1"'] & X \end{tikzcd} \tag{3.24}$$

*be a commutative square $\sigma$ of concentrated schemes and set $h = f_1 \circ g_1 = f_2 \circ g_2$. Let $u\colon X' \to X$ be any morphism and let $\sigma'$ be the fiber product of $\sigma$ with $X'$ over $X$, that is - the outer square $(Z', Y_1', Y_2', X')$ in*

*the commutative diagram*

$$\begin{tikzcd}
Z' \arrow[rrr, "g_2'"] \arrow[dr, "u"] \arrow[ddd, "g_1'"'] & & & Y_2' \arrow[dl, "u"'] \arrow[ddd, "f_2'"] \\
& Z \arrow[r, "g_2"] \arrow[d, "g_1"'] & Y_2 \arrow[d, "f_2"] & \\
& Y_1 \arrow[r, "f_1"'] & X & \\
Y_1' \arrow[ur, "u"] \arrow[rrr, "f_1'"'] & & & X' \arrow[ul, "u"']
\end{tikzcd} \tag{3.25}$$

*where* $Y_i' = Y_i \times_{X,f_i,u} X'$, $Z' = Z \times_{X,h,u} X' = Z \times_{Y_i,g_i,u} Y_i'$ *and the four squares between* $\sigma'$ *and* $\sigma$ *are all fiber squares. Then, setting* $h' = f_1' \circ g_1' = f_2' \circ g_2'$, *for any objects* $A_i \in D(Y_i)$*:*

(1) *The following diagram commutes in* $D(X')$*:*

$$\begin{tikzcd}
u^*\left(f_{1*}(A_1) \otimes f_{2*}(A_2)\right) \arrow[r, "u^*\kappa_\sigma"] \arrow[d, "\text{base change } u^* f_{i*} \to f'_{i*} u^*"'] & u^* h_* \left(g_1^*(A_1) \otimes g_2^*(A_2)\right) \arrow[d, "\text{base change } u^* h_* \to h'_* u^*"] \\
f'_{1*}(u^* A_1) \otimes f'_{2*}(u^* A_2) \arrow[r, "\kappa_{\sigma'}"'] & h'_* \left(g_1'^*(u^* A_1) \otimes g_2'^*(u^* A_2)\right)
\end{tikzcd} \tag{3.26}$$

(2) *The following diagram commutes in* $D(X)$*:*

$$\begin{tikzcd}
f_{1*}(A_1) \otimes f_{2*}(A_2) \arrow[r, "\kappa_\sigma"] \arrow[d, "\beta_u"'] & h_* \left(g_1^*(A_1) \otimes g_2^*(A_2)\right) \arrow[dd, "h_* \beta_u"] \\
u_* u^*(f_{1*}(A_1) \otimes f_{2*}(A_2)) \arrow[d, "u_*\left(\text{base change } u^* f_{i*} \to f'_{i*} u^*\right)"'] & \\
u_*(f'_{1*}(u^* A_1) \otimes f'_{2*}(u^* A_2)) \arrow[r, "u_* \kappa_{\sigma'}"'] & u_* h'_* \left(g_1'^*(u^* A_1) \otimes g_2'^*(u^* A_2)\right)
\end{tikzcd} \tag{3.27}$$

*where* $\beta_u$ *is the adjunction morphism* $\mathrm{Id} \to u_* u^*$.

*Proof.* The diagram (3.26) is the left adjoint of the diagram (3.27) with respect to $u_*$, so it suffices to only prove that (3.27) commutes. Let us take the left adjoint of the diagram (3.27) with respect to $u_* h'_*$. The left adjoint with respect to $u_* h'_*$ of the top half

$$f_{1*}A_1 \otimes f_{2*}A_2 \xrightarrow{\kappa_\sigma} h_* \left(g_1^* A_1 \otimes g_2^* A_2\right) \xrightarrow{h_* \gamma_u} h_* u_* \left(u^* g_1^* A_1 \otimes u^* g_2^* A_2\right) \xrightarrow{\sim} u_* h'^* \left(g_1'^* u^* A_1 \otimes g_2'^* u^* A_2\right)$$

of (3.27) is the composition of $h'^* u^* \left(f_{1*}A_1 \otimes f_{2*}A_2\right) \xrightarrow{\sim} u^* h^* \left(f_{1*}A_1 \otimes f_{2*}A_2\right)$ with the left adjoint of

$$f_{1*}(A_1) \otimes f_{2*}(A_2) \xrightarrow{\kappa_\sigma} h_* \left(g_1^*(A_1) \otimes g_2^*(A_2)\right) \xrightarrow{h_* \gamma_u} h_* u_* \left(u^* g_1^* A_1 \otimes u^* g_2^* A_2\right) \xrightarrow{\sim} h_* u_* \left(g_1'^* u^* A_1 \otimes g_2'^* u^* A_2\right)$$

with respect to $h_* u_*$. Making use of the definition of $\kappa_\sigma$ in (3.19), this adjoint works out to be

$$u^* h^* \left(f_{1*}A_1 \otimes f_{2*}A_2\right) \xrightarrow{\sim} \bigotimes_i u^* g_i^* f_i^* f_{i*} A_i \xrightarrow{\bigotimes_i u^* g_i^* \gamma_{f_i}} \bigotimes_i u^* g_i^* A_i \xrightarrow{\sim} \bigotimes_i g_i'^* u^* A_i$$

Composing with $h'^* u^* \left(f_{1*}A_1 \otimes f_{2*}A_2\right) \xrightarrow{\sim} u^* h^* \left(f_{1*}A_1 \otimes f_{2*}A_2\right)$ and expanding we can therefore write the left adjoint of the top half of (3.27) with respect to $u_* h'_*$ as

$$h'^* u^* \left(f_{1*}A_1 \otimes f_{2*}A_2\right) \xrightarrow{\sim}$$

$$\xrightarrow{\sim} \bigotimes_i g_i'^* f_i'^* u^* f_{i*} A_i \xrightarrow{\sim} \bigotimes_i g_i'^* u^* f_i^* f_{i*} A_i \xrightarrow{\sim} \bigotimes_i u^* g_i^* f_i^* f_{i*} A_i \xrightarrow{\bigotimes_i u^* g_i^* \gamma_{f_i}} \bigotimes_i u^* g_i^* A_i \xrightarrow{\sim} \bigotimes_i g_i'^* u^* A_i$$

which simplifies to

$$h'^*u^*\left(f_{1*}A_1 \otimes f_{2*}A_2\right) \xrightarrow{\sim} \bigotimes_i g_i'^* f_i'^* u^* f_{i*} A_i \xrightarrow{\sim} \bigotimes_i g_i'^* u^* f_i^* f_{i*} A_i \xrightarrow{\otimes_i g_i'^* u^* \gamma_{f_i}} \bigotimes_i g_i'^* u^* A_i$$

Similarly, the left adjoint of the bottom half

$$f_{1*}A_1 \otimes f_{2*}A_2 \xrightarrow{\beta_u} u_*u^*\left(f_{1*}A_1 \otimes f_{2*}A_2\right) \xrightarrow{\text{b.change}} u_*\left(f'_{1*}u^*A_1 \otimes f'_{2*}u^*A_2\right) \xrightarrow{u_*\kappa_{\sigma'}} u_*h'_*\left(g_1'^*u^*A_1 \otimes g'^*2u^*A_2\right)$$

of (3.27) with respect to $u_*h'_*$ works out as

$$h'^*u^*\left(f_{1*}A_1 \otimes f_{2*}A_2\right) \xrightarrow{\sim} \bigotimes_i g_i'^* f_i'^* u^* f_{i*} A_i \xrightarrow{\text{b.change}} \bigotimes_i g_i'^* f_i'^* f'_{i*} u^* A_i \xrightarrow{\otimes_i g_i'^* \beta_{f'_i}} \bigotimes_i g_i'^* u^* A_i$$

It therefore suffices to show that the following diagram commutes for $i = 1, 2$ and for all $A_i \in D(Y_i)$:

(3.28)
$$\begin{tikzcd}
f_i'^* u^* f_{i*} A_i \arrow[r, "\sim"] \arrow[d, "f_i'^*(u^* f_{i*} \to f'_{i*} u^*)"'] & u^* f_i^* f_{i*} A_i \arrow[d, "u^*\gamma_{f_i}"] \\
f_i'^* f'_{i*} u^* A_i \arrow[r, "\gamma_{f'_i}"'] & u^* A_i
\end{tikzcd}$$

But the base change morphism $u^*f_{i*} \to f'_{i*}u^*$ is defined as the right adjoint with respect to $f_i'^*$ of precisely the composition $f_i'^*u^*f_{i*} \xrightarrow{\sim} u^*f_i^*f_{i*} \xrightarrow{u^*\gamma_{f_i}} u^*$, so the right adjoint of (3.28) with respect to $f_i'^*$ is the diagram

(3.29)
$$\begin{tikzcd}
u^* f_{i*} A_i \arrow[d, "u^* f_{i*} \to f'_{i*} u^*"'] \arrow[dr, "u^* f_{i*} \to f'_{i*} u^*"] & \\
f'_{i*} u^* A_i \arrow[r, "\mathrm{Id}"'] & f'_{i*} u^* A_i
\end{tikzcd}$$

which clearly commutes. □

The reader may now check that if we apply Proposition 3.4 to the context of Section 3.1 by setting $X = X_1 \times X_2 \times X_1$, $Y_1 = D_{12}$, $Y_2 = D_{23}$, $Z = D'$, $A_1 = \mathcal{O}_{D_{12}}$, $A_2 = \wedge^d \mathcal{N}_{23}|_{D_{23}}[d]$ and setting $X' \xrightarrow{u} X$ to be the diagonal inclusion $X_1 \times X_2 \xrightarrow{\Delta} X_1 \times X_2 \times X_1$, then the bottom half

$$f_{1*}(A_1) \otimes f_{2*}(A_2) \xrightarrow{\beta_u} u_*u^*(f_{1*}(A_1) \otimes f_{2*}(A_2)) \xrightarrow{\sim} u_*(f'_{1*}(u^*A_1) \otimes f'_{2*}(u^*A_2)) \xrightarrow{u_*\kappa_{\sigma'}} u_*h'_*\left(g_1'^*(u^*A_1) \otimes g_2'^*(u^*A_2)\right)$$

of the diagram (3.27) is precisely the composition

$$\mathcal{O}_{D_{12}} \otimes \mathcal{O}_{D_{23}}^\vee \to \Delta_*\Delta^*\left(\mathcal{O}_{D_{12}} \otimes \mathcal{O}_{D_{23}}^\vee\right) \simeq \Delta_*\left(\mathcal{O}_D \otimes \mathcal{O}_D^\vee\right) \to \Delta_*\mathcal{O}_D^\vee$$

of the derived restriction (3.9) to the diagonal in $X_1 \times X_2 \times X_1$ followed by the projection $\Delta_* \bigoplus_{i=0}^d \wedge^i \mathcal{N}|_D[i] \to \Delta_* \wedge^d \mathcal{N}|_D[d]$ while the top half

$$f_{1*}(A_1) \otimes f_{2*}(A_2) \xrightarrow{\kappa_\sigma} h_*\left(g_1^*(A_1) \otimes g_2^*(A_2)\right) \xrightarrow{h_*\beta_u} h_*u_*u^*\left(g_1^*(A_1) \otimes g_2^*(A_2)\right)$$

of the diagram (3.27) is precisely the composition

$$\mathcal{O}_{D_{12}} \otimes \mathcal{O}_{D_{23}}^\vee \xrightarrow{\sim} \wedge^d \mathcal{N}_{23}|_{D'} \to \wedge^d \mathcal{N}_{23}|_{\Delta D}$$

of an identification with an object on $D'$ followed by the derived restriction to the diagonal in $D'$.

3.4. **The adjunction counit for the pushforward Fourier-Mukai kernels.** We can now apply the generalities of the previous two sections to obtain an alternative decomposition to that in Theorem 2.1 of the morphism of Fourier-Mukai kernels which induces the canonical adjunction morphism $\Phi_E^{ladj}\Phi_E \to \mathrm{Id}$ in case where $E$ is a pushforward of an object on some $D \hookrightarrow X_1 \times X_2$.

Let $X_1$ and $X_2$ be a pair of separable schemes of finite type over a field $k$. Let $D \xrightarrow{\iota_D} X_1 \times X_2$ be a closed immersion proper over both $X_1$ and $X_2$. Denote by $\pi_{D1}$ the composition $D \xrightarrow{\iota_D} X_1 \times X_2 \xrightarrow{\pi_1} X_1$. Let $D_{12}$ be defined by the following fiber square:

(3.30)
$$\begin{tikzcd} D_{12} \arrow[r, "\iota_{D12}"] \arrow[d, "\pi_{D12}"'] & X_1 \times X_2 \times X_1 \arrow[d, "\pi_{12}"] \\ D \arrow[r, "\iota_D"'] & X_1 \times X_2 \end{tikzcd}$$

and similarly for $D_{23}$. Then $D' = D_{12} \cap D_{23} \xrightarrow{\iota_{D'}} X_1 \times X_2 \times X_1$ fits into the following fiber square:

(3.31)
$$\begin{tikzcd} D' \arrow[r, "\iota'_{23}"] \arrow[d, "\iota'_{12}"'] \arrow[dr, "\iota_{D'}"] & D_{23} \arrow[d, "\iota_{D23}"] \\ D_{12} \arrow[r, "\iota_{D12}"'] & X_1 \times X_2 \times X_1 \end{tikzcd}$$

and the map $X_1 \times X_2 \xrightarrow{\Delta} X_1 \times X_2 \times X_1$ defines the following base change diagram for (3.31):

(3.32)
$$\begin{tikzcd} D \arrow[rrr, "\mathrm{Id}"] \arrow[dr, "\Delta"] \arrow[ddd, "\mathrm{Id}"'] & & & D \arrow[dl, "\Delta"'] \arrow[ddd, "\iota_D"] \\ & D' \arrow[r, "\iota'_{23}"] \arrow[d, "\iota'_{12}"'] \arrow[dr, "\iota_{D'}"] & D_{23} \arrow[d, "\iota_{D23}"] & \\ & D_{12} \arrow[r, "\iota_{D12}"'] & X_1 \times X_2 \times X_1 & \\ D \arrow[rrr, "\iota_D"'] \arrow[ur, "\Delta"] & & & X_1 \times X_2 \arrow[ul, "\Delta"'] \end{tikzcd}$$

Let $E_D$ be a perfect object in $D(D)$ such that $E = \iota_{D*}(E_D)$ is perfect in $D(X_1 \times X_2)$. Let $E_{12} = \pi^*_{D12}(E_D)$, $E'_{12} = \iota'^*_{12}E_{12}$, $E_{23} = \pi^*_{D23}E_D$ and $(E^\vee_{23})' = \iota'^*_{23}(E^\vee_{23})$.

**Theorem 3.1.** *The adjunction morphism $\Phi_E^{ladj}\Phi_E \to \mathrm{Id}_{X_1}$ is isomorphic to the morphism of Fourier-Mukai transforms induced by the morphism of objects of $D(X_1 \times X_1)$ which is the composition of the following:*

$$Q = \pi_{13*}\left(\iota_{D12*}E_{12} \otimes \iota_{D23*}\left(E^\vee_{23} \otimes \pi^*_{D23}\pi^!_{D1}(\mathcal{O}_{X_1})\right)\right) \to \pi_{13*}\iota_{D'*}\left(E'_{12} \otimes (E^\vee_{23})' \otimes \iota'^*_{23}\pi^*_{D23}\pi^!_{D1}(\mathcal{O}_{X_1})\right) \tag{3.33}$$
$$\pi_{13*}\iota_{D'*}\left(E'_{12} \otimes (E^\vee_{23})' \otimes \iota'^*_{23}\pi^*_{D23}\pi^!_{D1}(\mathcal{O}_{X_1})\right) \to \pi_{13*}\iota_{D'*}\Delta_*\Delta^*\left(E'_{12} \otimes (E^\vee_{23})' \otimes \iota'^*_{23}\pi^*_{D23}\pi^!_{D1}(\mathcal{O}_{X_1})\right) \tag{3.34}$$
$$\pi_{13*}\iota_{D'*}\Delta_*\Delta^*\left(E'_{12} \otimes (E^\vee_{23})' \otimes \iota'^*_{23}\pi^*_{D23}\pi^!_{D1}(\mathcal{O}_{X_1})\right) \simeq \Delta_*\pi_{D1*}\left(E_D \otimes E^\vee_D \otimes \pi^!_{D1}(\mathcal{O}_{X_1})\right) \tag{3.35}$$
$$\Delta_*\pi_{D1*}\left(E_D \otimes E^\vee_D \otimes \pi^!_{D1}(\mathcal{O}_{X_1})\right) \to \Delta_*\pi_{D1*}\pi^!_{D1}(\mathcal{O}_{X_1}) \tag{3.36}$$
$$\Delta_*\pi_{D1*}\pi^!_{D1}(\mathcal{O}_{X_1}) \to \Delta_*\mathcal{O}_{X_1} \tag{3.37}$$

*where (3.33) is induced by the Künneth map for the fiber square (3.31), (3.34) is induced by the adjunction unit $\mathrm{Id}_{D(D')} \to \Delta_*\Delta^*$ (i.e. the derived restriction to diagonal $\Delta D$ in $D'$), the isomorphism (3.35) is due to scheme map identities $\Delta \circ \iota_{D'} = \iota_D \circ \Delta$, $\Delta \circ \pi_{13} = \pi_1 \circ \Delta$ and $\Delta \circ \iota'_{23} \circ \pi_{D23} = \Delta \circ \iota'_{12} \circ \pi_{D12} = \mathrm{Id}_D$, (3.36) is induced by evaluation map $ev_{E_D}$ on $D$ and (3.37) is induced by the adjunction counit $\pi_{D1*}\pi^!_{D1} \to \mathrm{Id}_{D(X_1)}$.*

*Proof.* Assume first that $X_2$ is proper. Then by Theorem 2.1 the adjunction morphism $\Phi_E^{ladj}\Phi_E \to \mathrm{Id}_{X_1}$ is induced by the composition

$$\pi_{13*}\left(\pi_{12}^* E \otimes \pi_{23}^* E^\vee \otimes \pi_{23}^*\pi_1^!(\mathcal{O}_{X_1})\right) \to \pi_{13*}\Delta_*\Delta^*\left(\pi_{12}^* E \otimes \pi_{23}^* E^\vee \otimes \pi_{23}^*\pi_1^!(\mathcal{O}_{X_1})\right) \tag{3.38}$$

$$\pi_{13*}\Delta_*\Delta^*\left(\pi_{12}^* E \otimes \pi_{23}^* E^\vee \otimes \pi_{23}^*\pi_1^!(\mathcal{O}_{X_1})\right) \quad\simeq\quad \Delta_*\pi_{1*}\left(E \otimes E^\vee \otimes \pi_1^!(\mathcal{O}_{X_1})\right) \tag{3.39}$$

$$\Delta_*\pi_{1*}\left(E \otimes E^\vee \otimes \pi_1^!(\mathcal{O}_{X_1})\right) \to \Delta_*\pi_{1*}\left(\pi_1^!(\mathcal{O}_{X_1})\right) \tag{3.40}$$

$$\Delta_*\pi_{1*}\left(\pi_1^!(\mathcal{O}_{X_1})\right) \to \Delta_*\mathcal{O}_{X_1}. \tag{3.41}$$

By the flat base change we have $\pi_{12}^*E = \pi_{12}^*\iota_{D*}E_D \simeq \iota_{D12*}\pi_{D12}^*E_D \simeq \iota_{D12*}E_{12}$. Similarly, by the flat base change and by the Grothendieck duality we have:

$$\begin{aligned} &\pi_{23}^*\left(E^\vee \otimes \pi_1^!\mathcal{O}_{X_1}\right) \simeq \pi_{23}^*\,\mathbf{R}\mathcal{H}om\left(\iota_{D*}E_D, \pi_1^!\mathcal{O}_{X_1}\right) \simeq \pi_{23}^*\iota_{D*}\,\mathbf{R}\mathcal{H}om\left(E_D, \pi_{D1}^!\mathcal{O}_{X_1}\right) \simeq \\ \simeq\quad &\iota_{D23*}\pi_{D23}^*\,\mathbf{R}\mathcal{H}om\left(E_D, \pi_{D1}^!\mathcal{O}_{X_1}\right) \simeq \iota_{D23*}\,\mathbf{R}\mathcal{H}om\left(E_{23}, \pi_{D23}^*\pi_{D1}^!\mathcal{O}_{X_1}\right) \simeq \iota_{D23*}\left(E_{23}^\vee \otimes \pi_{D23}^*\pi_{D1}^!\mathcal{O}_{X_1}\right)\end{aligned}$$

We can therefore re-write (3.38)-(3.39) as the composition

$$\pi_{13*}\left(\pi_{12}^* E \otimes \pi_{23}^* E^\vee \otimes \pi_{23}^*\pi_1^!(\mathcal{O}_{X_1})\right) \xrightarrow{\sim} \pi_{13*}\left(\iota_{D12*}E_{12} \otimes \iota_{D23*}\left(E_{23}^\vee \otimes \pi_{D23}^*\pi_{D1}^!\mathcal{O}_{X_1}\right)\right) \tag{3.42}$$

$$\pi_{13*}\left(\iota_{D12*}E_{12} \otimes \iota_{D23*}\left(E_{23}^\vee \otimes \pi_{D23}^*\pi_{D1}^!\mathcal{O}_{X_1}\right)\right) \to \pi_{13*}\Delta_*\Delta^*\left(\text{——''——}\right) \tag{3.43}$$

$$\pi_{13*}\Delta_*\Delta^*\left(\iota_{D12*}E_{12} \otimes \iota_{D23*}\left(E_{23}^\vee \otimes \pi_{D23}^*\pi_{D1}^!\mathcal{O}_{X_1}\right)\right) \xrightarrow{\sim} \pi_{13*}\Delta_*\left(\iota_{D*}E \otimes \iota_{D*}\left(E_D^\vee \otimes \pi_{D1}^!\mathcal{O}_{X_1}\right)\right) \tag{3.44}$$

$$\pi_{13*}\Delta_*\left(\iota_{D*}E_D \otimes \iota_{D*}\left(E_D^\vee \otimes \pi_{D1}^!\mathcal{O}_{X_1}\right)\right) \xrightarrow{\sim} \Delta_*\pi_{1*}\left(E \otimes E^\vee \otimes \pi_1^!\mathcal{O}_{X_1}\right) \tag{3.45}$$

In (3.44) we first use scheme map identity $\Delta\circ\pi_{13} = \pi_1\circ\Delta$, then base change isomorphisms $\Delta^*\iota_{D12*} \to \iota_{D*}\Delta^*$ and $\Delta^*\iota_{D23*} \to \iota_{D*}\Delta^*$ which we have since the corresponding fiber squares are clearly Tor-independent and then we use scheme map identities $\Delta\circ\pi_{D12} = \Delta\circ\pi_{D23} = \mathrm{Id}_D$ to write $\Delta^*E_{12} \simeq E_D$ and $\Delta^*E_{23}^\vee = E_D^\vee$. In (3.45) we use the Grothendieck duality again:

$$\iota_{D*}\left(E_D^\vee \otimes \pi_{D1}^!\mathcal{O}_{X_1}\right) \simeq \iota_{D*}\,\mathbf{R}\mathcal{H}om\left(E_D, \iota_D^!\pi_1^!\mathcal{O}_{X_1}\right) \simeq \mathbf{R}\mathcal{H}om\left(\iota_{D*}E_D, \pi_1^!\mathcal{O}_{X_1}\right) \simeq E^\vee \otimes \pi_1^!\mathcal{O}_{X_1}.$$

We now apply Proposition 3.1 to decompose (3.40) as

$$\begin{aligned} &\Delta_*\pi_{1*}\left(E \otimes E^\vee \otimes \pi_1^!(\mathcal{O}_{X_1})\right) \xrightarrow{\sim} \Delta_*\pi_{1*}\left(\iota_{D*}E_D \otimes \iota_{D*}(E_D^\vee \otimes \iota_D^!\pi_1^!(\mathcal{O}_{X_1})\right) \to \\ \to\ &\Delta_*\pi_{1*}\iota_{D*}\left(E_D \otimes E_D^\vee \otimes \iota_D^!\pi_1^!(\mathcal{O}_{X_1})\right) \to \Delta_*\pi_{1*}\iota_{D*}\iota_D^!\pi_1^!(\mathcal{O}_{X_1}) \to \Delta_*\pi_{1*}\pi_1^!(\mathcal{O}_{X_1})\end{aligned}$$

We conclude that we can re-write (3.38)-(3.41) as a composition of the isomorphism

$$\pi_{13*}\left(\pi_{12}^* E \otimes \pi_{23}^* E^\vee \otimes \pi_{23}^*\pi_1^!(\mathcal{O}_{X_1})\right) \xrightarrow{\sim} \pi_{13*}\left(\iota_{D12*}E_{12} \otimes \iota_{D23*}\left(E_{23}^\vee \otimes \pi_{D23}^*\pi_{D1}^!\mathcal{O}_{X_1}\right)\right)$$

with

$$\pi_{13*}\left(\iota_{D12*}E_{12} \otimes \iota_{D23*}\left(E_{23}^\vee \otimes \pi_{D23}^*\pi_{D1}^!\mathcal{O}_{X_1}\right)\right) \to \pi_{13*}\Delta_*\Delta^*\left(\text{——''——}\right) \tag{3.46}$$

$$\pi_{13*}\Delta_*\Delta^*\left(\iota_{D12*}E_{12} \otimes \iota_{D23*}\left(E_{23}^\vee \otimes \pi_{D23}^*\pi_{D1}^!\mathcal{O}_{X_1}\right)\right) \xrightarrow{\sim} \pi_{13*}\Delta_*\left(\iota_{D*}E_D \otimes \iota_{D*}\left(E_D^\vee \otimes \pi_{D1}^!\mathcal{O}_{X_1}\right)\right) \tag{3.47}$$

$$\pi_{13*}\Delta_*\left(\iota_{D*}E_D \otimes \iota_{D*}\left(E_D^\vee \otimes \pi_{D1}^!\mathcal{O}_{X_1}\right)\right) \to \pi_{13*}\Delta_*\iota_{D*}\left(E_D \otimes E_D^\vee \otimes \iota_D^!\pi_1^!(\mathcal{O}_{X_1})\right) \tag{3.48}$$

$$\pi_{13*}\Delta_*\iota_{D*}\left(E_D \otimes E_D^\vee \otimes \iota_D^!\pi_1^!(\mathcal{O}_{X_1})\right) \xrightarrow{\sim} \Delta_*\pi_{1*}\iota_{D*}\left(E_D \otimes E_D^\vee \otimes \iota_D^!\pi_1^!(\mathcal{O}_{X_1})\right) \tag{3.49}$$

$$\Delta_*\pi_{1*}\iota_{D*}\left(E_D \otimes E_D^\vee \otimes \iota_D^!\pi_1^!(\mathcal{O}_{X_1})\right) \to \Delta_*\pi_{1*}\iota_{D*}\iota_D^!\pi_1^!(\mathcal{O}_{X_1}) \tag{3.50}$$

$$\Delta_*\pi_{1*}\iota_{D*}\iota_D^!\pi_1^!(\mathcal{O}_{X_1}) \to \Delta_*\pi_{1*}\pi_1^!(\mathcal{O}_{X_1}) \tag{3.51}$$

$$\Delta_*\pi_{1*}\pi_1^!(\mathcal{O}_{X_1}) \to \Delta_*\mathcal{O}_{X_1} \tag{3.52}$$

The claim of the theorem now follows - the adjunction counits for $\iota_{D*}$ and $\pi_{1*}$ in (3.51) and (3.52) compose to give the adjunction counit for $\pi_{D1*}$ in (3.37), while the composition (3.46)-(3.49) of the restriction to the diagonal in $X_1\times X_2\times X_1$ followed by the Künneth map for the outer square of (3.32) equals by the base change for Künneth maps (Prop. 3.4(2)) to the composition (3.33)-(3.35) of the Künneth map for the inner square of (3.32) followed by the restriction to the diagonal in $D'$.

Suppose now $X_2$ is not proper. Then, following Section 2.2, we compactify $X_2$ by choosing an open immersion $X_2 \to \bar{X}_2$ with $\bar{X}_2$ proper. We also make use of the notation introduced in Section 2.2 where compactified versions of various objects and morphisms are denoted by the bar above them. E.g. we denote

the inclusion $D \xrightarrow{\iota_D} X_1 \times X_2 \xrightarrow{\iota} X_1 \times \bar{X}_2$ by $\bar{\iota}_D$. Then by the argument above the composition of the compactified versions of (3.33) - (3.37) induces the compactified adjunction counit $\Phi^{ladj}_{\bar{E}} \Phi_{\bar{E}} \to \mathrm{Id}_{X_1}$. By the results of Section 2.2 the compactified and the uncompactified adjunction counits are naturally isomorphic. Since we need the morphism $Q \to \mathcal{O}_{X_1}$ to induce the adjunction counit $\Phi^{ladj}_{E} \Phi_{E} \to \mathrm{Id}_{D(X_1)}$ only up to an isomorphism, we need only to show it to be isomorphic to the compactified (3.33)-(3.37). To prove the claim of the theorem it suffices therefore to exhibit an isomorphism $\bar{Q} \xrightarrow{\sim} Q$ which composed with the uncompactified (3.33)-(3.37) gives the compactified (3.33)-(3.37).

Observe now that (3.34)-(3.37) are actually completely independent of the ambient space $X_2$ and so their compactified versions are identical to the uncompactified ones. It therefore suffices to find an isomorphism $\bar{Q} \xrightarrow{\sim} Q$ that would make the following diagram commute:

(3.53)
$$\begin{tikzcd} Q \arrow[dr, "\text{uncompact. (3.33)}"] & & \\ \bar{Q} \arrow[u, "\sim"] \arrow[r, "\text{compact. (3.33)}"'] & \pi_{13*}\iota_{D'*}\left(E'_{12} \otimes (E^\vee_{23})' \otimes \iota'^*_{23}\pi^*_{D23}\pi^!_{D1}(\mathcal{O}_{X_1})\right) \arrow[r, "(3.34)-(3.37)"'] & \Delta_*\mathcal{O}_{X_1} \end{tikzcd}$$

But the compactified (3.33) is the image under $\bar{\pi}_{13*}$ of the Künneth map for the square

(3.54)
$$\begin{tikzcd} D' \arrow[r, "\iota'_{23}"] \arrow[d, "\iota'_{12}"'] \arrow[dr, "\iota_{D'}"] & D_{23} \arrow[d, "\iota_{D23}"] \\ D_{12} \arrow[r, "\iota_{D12}"'] & X_1 \times \bar{X}_2 \times X_1 \end{tikzcd}$$

and the uncompactified (3.33) is the image under $\pi_{13} = \bar{\pi}_{13*}\iota_*$ of the Künneth map for the square obtained from (3.54) by the base change $\iota\colon X_1 \times X_2 \times X_1 \to X_1 \times \bar{X}_2 \times X_1$. The desired statement is then precisely the base change for Künneth maps (Prps. 3.4): the diagram (3.53) is obtained by applying $\bar{\pi}_{13*}$ to the commutative diagram in (3.27). $\square$

As before, it is worth spelling out the parallel statement for the right adjunction counit. So with the same notation as above let $\Psi_E$ be the Fourier-Mukai transform from $D(X_2)$ to $D(X_1)$ with the kernel $E$ and let $\Psi^{\mathrm{radj}}_E$ be its right adjoint.

**Corollary 3.5.** *The adjunction counit $\Psi_E \Psi^{radj}_E \to \mathrm{Id}_{X_1}$ is isomorphic to the morphism of Fourier-Mukai transforms induced by the following morphism of objects of $D(X_1)$:*

$$Q = \pi_{13*}\left(\iota_{D12*}\left(E^\vee_{12} \otimes \pi^*_{D12}\pi^!_{D1}\mathcal{O}_{X_1}\right) \otimes \iota_{D23*}E_{23}\right) \to \pi_{13*}\iota_{D'*}\left((E^\vee_{12})' \otimes E_{23} \otimes \iota'^*_{12}\pi^*_{D12}\pi^!_{D1}\mathcal{O}_{X_1}\right) \tag{3.55}$$

$$\pi_{13*}\iota_{D'*}\left((E^\vee_{12})' \otimes E_{23} \otimes \iota'^*_{12}\pi^*_{D12}\pi^!_{D1}\mathcal{O}_{X_1}\right) \to \pi_{13*}\iota_{D'*}\Delta_*\Delta^*\left((E^\vee_{12})' \otimes E_{23} \otimes \iota'^*_{12}\pi^*_{D12}\pi^!_{D1}\mathcal{O}_{X_1}\right) \tag{3.56}$$

$$\pi_{13*}\iota_{D'*}\Delta_*\Delta^*\left((E^\vee_{12})' \otimes E_{23} \otimes \iota'^*_{12}\pi^*_{D12}\pi^!_{D1}\mathcal{O}_{X_1}\right) \simeq \Delta_*\pi_{D1*}\left(E_D \otimes E^\vee_D \otimes \pi^!_{D1}\mathcal{O}_{X_1}\right) \tag{3.57}$$

$$\Delta_*\pi_{D1*}\left(E_D \otimes E^\vee_D \otimes \pi^!_{D1}\mathcal{O}_{X_1}\right) \to \Delta_*\pi_{D1*}\pi^!_{D1}\mathcal{O}_{X_1} \tag{3.58}$$

$$\Delta_*\pi_{D1*}\pi^!_{D1}\mathcal{O}_{X_1} \to \Delta_*\mathcal{O}_{X_1} \tag{3.59}$$

## 4. An example

Let us give a concrete example of using the results of section 3 to compute a twist of a Fourier-Mukai transform. For this example we choose the naive derived category transform induced by the Mukai flop. This transform is not an equivalence - it was proved by Namikawa in [Nam03] by direct comparison of Hom spaces. It therefore has a non-trivial twist - and below we show how Theorem 3.1 can be applied to compute the kernel which defines it as the Fourier-Mukai transform. We stress that the value of this section lies not in the answer itself, but in demonstrating how the methods of the paper apply to obtain it. However, the reader may observe that the kernel we obtain possesses a certain symmetry - and we shall demonstrate in [AL] that this symmetry is precisely the underlying reason for the braiding which occurs between natural spherical twists in derived categories of the cotangent bundles of complete flag varieties (see [KT07], §4).

Let $V$ be a 3-dimensional vector space and let $X_1$ be the scheme $T^*\mathbb{P}(V)$, that is - the total space of the cotangent bundle of $\mathbb{P}(V)$. Similarly, let $X_2$ be the scheme $T^*\mathbb{P}(V^\vee)$. These schemes admit a following description:

$$X_1 = \left\{ \; 0 \subset U \subset V \;, \; \alpha \in \mathrm{End}(V) \;\middle|\; \dim U = 1, \alpha(V) \subseteq U, \; \alpha(U) = 0 \; \right\}$$

$$X_2 = \left\{ \; 0 \subset W \subset V \;, \; \alpha \in \mathrm{End}(V) \;\middle|\; \dim W = 2, \alpha(V) \subseteq W, \; \alpha(W) = 0 \; \right\}$$

We have also a variety

$$D = \left\{ \; 0 \subset U \subset W \subset V \;, \; \alpha \in \mathrm{End}(V) \;\middle|\; \dim U = 1, \dim W = 2, \; \alpha(V) \subseteq U, \; \alpha(W) = 0 \; \right\}$$

Observe that there exist natural "forgetful" maps $\phi_k \colon D \to X_k$ which forget the subspace of dimension $3-k$. Each map $\phi_k$ is isomorphic to the blow-up of the zero section $\mathbb{P}^2$ carved out by $\alpha = 0$ in $X_k \simeq T^*\mathbb{P}^2$.

Let $\Phi$ be the functor $\phi_{2*}\phi_1^*$ from $D(X_1)$ to $D(X_2)$ and let us compute its left twist. The functor $\Phi$ is a Fourier-Mukai transform with the kernel $\iota_{D*}\mathcal{O}_D$, where $\iota_D = \phi_1 \times \phi_2 : D \to X_1 \times X_2$. Note that maps $\phi_k$ are proper and moreover, since each map $\phi_k$ is a blowup of $X_k$, we have $\phi_{k*}\mathcal{O}_D = \mathcal{O}_{X_k}$. Let $\pi_{ij} : X_1 \times X_2 \times X_1 \to X_i \times X_j$ be the natural projections, and define $D_{ij} = \pi_{ij}^{-1}(D)$. We have:

$$X_1 \times X_2 \times X_1 = \left\{ \; 0 \subset U_1, W_2, U_3 \subset V \;, \alpha_i \in \mathrm{End}(V) \;\middle|\; \begin{array}{c} \dim U_i = 1, \; \dim W_i = 2, \\ \alpha_i(V) \subseteq U_i \text{ or } W_i, \; \alpha_i(U_i \text{ or } W_i) = 0 \end{array} \right\}$$

with arrows labelled $\alpha_{1,2,3}$;

$$D_{12} = \left\{ \; 0 \subset U_1 \subset W_2 \subset V \;, \; 0 \subset U_3 \subset V \; \right\}$$

with arrows labelled $\alpha_1 = \alpha_2$ on the first chain and $\alpha_3$ on the second;

$$D_{23} = \left\{ \; 0 \subset U_1 \subset V \;, \; 0 \subset U_3 \subset W_2 \subset V \; \right\}$$

with arrows labelled $\alpha_1$ on the first chain and $\alpha_2 = \alpha_3$ on the second.

It follows that $D' = D_{12} \cap D_{23}$ can be described as

$$D' = \left\{ \; 0 \subset U_1, U_3 \subset W_2 \subset V \;, \; \alpha \in \mathrm{End}(V) \;\middle|\; \dim U_i = 1, \; \dim W_i = 2, \; \alpha(V) \subseteq U_1 \cap U_3, \; \alpha(W) = 0 \; \right\}$$

Observe that for any point of $D'$ we have $U_1 = U_3$ or $\alpha = 0$ (or both). Therefore $D'$ consists of two irreducible components: the diagonal

$$\Delta D = \{U_1 = U_3\} \subseteq D'$$

and the zero section

$$P = \{\alpha = 0\} \subseteq D'.$$

The component $P$ can be described as

$$P = \left\{ \; 0 \subset U_1, U_3 \subset W_2 \subset V \;\middle|\; \dim U_i = 1, \; \dim W_i = 2 \; \right\}$$

Let $\phi_{13}$ be the forgetful map $P \to \mathbb{P}(V) \times \mathbb{P}(V)$ which forgets the subspace $W_2$. Observe that $\phi_{13}$ is precisely the blowup of the diagonal $\Delta\mathbb{P}(V)$ in $\mathbb{P}(V) \times \mathbb{P}(V)$.

By the Corollary 1.2 of Theorem 3.1 the left twist of $\Phi$ is the Fourier-Mukai transform $X_1 \to X_1$ with the kernel $K = \pi_{13*}\iota_{D'*}\mathcal{I}_\Delta[-1] \in D(X_1 \times X_1)$, where $\iota_{D'}$ is the inclusion $D' \hookrightarrow X_1 \times X_2 \times X_1$ and $\mathcal{I}_\Delta$ is the ideal sheaf of $\Delta D$ in $D'$. Since $D'$ has two irreducible components $\Delta D$ and $P$, the sheaf $\mathcal{I}_\Delta$ is $\iota_{P*}\mathcal{I}_{\Delta D \cap P}$ where $\iota_P$ is the inclusion $P \hookrightarrow D'$ and $\mathcal{I}_{\Delta D \cap P}$ is the ideal sheaf of the intersection $(\Delta D) \cap P$ on $P$. We therefore have

$K = \pi_{13*}\iota_{D'*}\iota_{P*}\mathcal{I}_{\Delta D \cap P}[1]$. The intersection $(\Delta D) \cap P$ is carved out by $U_1 = U_3$ in $P$, i.e. it is the pre-image of the diagonal of $\mathbb{P}(V) \times \mathbb{P}(V)$ under $\phi_{13}$. Now observe that the following diagram commutes

$$\begin{tikzcd} P \arrow[r, "\iota_P"] \arrow[d, "\phi_{13}"'] & D' \arrow[r, "\iota_{D'}"] & X_1 \times X_2 \times X_1 \arrow[d, "\pi_{13}"'] \\ \mathbb{P}(V) \times \mathbb{P}(V) \arrow[rr, "\iota_0"'] & & X_1 \times X_1 \end{tikzcd}$$

where $\iota_0$ is the zero-section inclusion of $\mathbb{P}(V) \times \mathbb{P}(V)$ into $T^*\mathbb{P}(V) \times T^*\mathbb{P}(V)$. We conclude that

$$K \simeq \pi_{13*}\iota_{D'*}\iota_{P*}\mathcal{I}_{\Delta D \cap P}[1] \simeq \iota_{0*}\phi_{13*}\phi_{13}^*\mathcal{I}_{\Delta \mathbb{P}(V)} \simeq \iota_{0*}\, \mathcal{I}_{\Delta \mathbb{P}(V)}[1] = \iota_{0*}\, \mathcal{O}_{\mathbb{P}(V)\times\mathbb{P}(V)}(-1,-1)$$

Here we used the fact that a derived pushdown down a blowup of the pullback of the blown up ideal is that ideal itself.

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